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SUPERCONVERGENCE OF GRADIENT RECOVERY SCHEMES ON GRADED MESHES FOR CORNER SINGULARITIES*

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Abstract

For the linear finite element solution to the Poisson equation, we show that superconvergence exists for a type of graded meshes for corner singularities in polygonal domains. In particular, we prove that the L^2-projection from the piecewise constant field ∇u_N to the continuous and piecewise linear finite element space gives a better approximation of ∇u in the H^1-norm. In contrast to the existing superconvergence results, we do not assume high regularity of the exact solution.

Mathematics subject classification: 65N12, 65N30, 65N50.
Key words: Superconvergence, Graded meshes, Weighted Sobolev spaces, Singular solutions, The finite element method, Gradient recovery schemes.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. We shall consider the linear finite element approximation for the Poisson equation

$$-\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega. \tag{1.1}$$

We are interested in the case when Ω is concave, and thus the solution of (1.1) possesses corner singularities at vertices of Ω where some of the interior angles are greater than π.

By the regularity theory, the solution u is in $H^{1+\beta}(\Omega)$ with $\beta = \min_i\{\pi/\alpha_i, 1\}$, where α_i are interior angles of the polygonal domain Ω. It is easy to see that when the maximum angle is larger than π, i.e., Ω is concave, $u \notin H^2(\Omega)$, and thus the finite element approximation based on quasi-uniform grids will not produce the optimal convergence rate. Graded meshes near the singular vertices are employed to recovery the optimal convergence rate. Such meshes can be constructed based on a priori estimates [3,4,6,24,25,31,37] or on a posteriori analysis [9,12,39]. In this paper, we shall consider the approach used in [6,31], and in particular, focus on the linear finite element approximation of (1.1).

In [6,31], a sequence of linear finite element spaces $\mathbb{V}_N \subset H_0^1(\Omega)$ is constructed, such that

$$\|\nabla(u - u_N)\|_{L^2(\Omega)} \le CN^{-1/2}\|f\|_{L^2(\Omega)}, \quad \forall f \in L^2(\Omega), \tag{1.2}$$

where $u_N = u_{\mathbb{V}_N}$ is the finite element approximation and $N = \dim \mathbb{V}_N$. The convergence rate $N^{-1/2}$ in (1.2) is the best possible rate we can expect for the linear element, and the solution

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u_N is the best approximation (i.e., the projection) of u into $\mathbb{V}_N$ in the H^1 semi-norm. We cannot find a better approximation to u in the space $\mathbb{V}_N$ measured in the H^1 semi-norm.

The main contribution of this paper is to demonstrate that appropriate post-processing of the piecewise constant vector function ∇u_N will improve the convergence rate. More precisely, let $\overline{\mathbb{V}}_N$ denote the space of continuous and piecewise linear finite element functions. Note that $\overline{\mathbb{V}}_N$ is bigger than $\mathbb{V}_N$ since it also contains nodal basis of boundary nodes. For any $u \in L^2(\Omega)$, denote by

$$Q_N : L^2(\Omega) \mapsto \overline{\mathbb{V}}_N, \quad (Q_N u, v_n)_{L^2} := (u, v_n)_{L^2}, \quad \forall v_n \in \overline{\mathbb{V}}_N,$$

the L^2-projection to $\overline{\mathbb{V}}_N$, and for $u \in H^1(\Omega)$,

$$Q_N(\nabla u) := Q_N(\partial_x u, \partial_y u) = (Q_N(\partial_x u), Q_N(\partial_y u)) \in \overline{\mathbb{V}}_N \times \overline{\mathbb{V}}_N.$$

Then on appropriate graded meshes and for any $\delta > 0$, we shall show

$$\|\nabla u - Q_N(\nabla u_N)\|_{L^2(\Omega)} \le C N^{-5/8+\delta} \|f\|_{H^1(\Omega)}, \quad \forall f \in H^1(\Omega), \tag{1.3}$$

where C depends only on the interior angles of Ω, the initial triangulation $\mathcal{T}_0$ of Ω, and the constant δ. Therefore, we obtain a better approximation of ∇u based on existing information on the mesh and corresponding matrices. Note that instead of the inversion of the stiffness matrix, the computation of $Q_N(\nabla u_N)$ only involves the inversion of the mass matrix. Following our diagonal scaling technique in Section 2, the preconditioned conjugate gradient (PCG) method with the diagonal pre-conditioner will be convergent very quickly. Consequently, the computational cost of $Q_N u_N$ is negligible comparing with that of u_N.

The improved convergence rate (1.3) is known as superconvergence in the literature. Let $u_I \in \mathbb{V}_N$ be the nodal interpolation of u. Our proof of (1.3) is based on the following super-closeness between u_N and u_I in $\mathbb{V}_N$:

$$\|\nabla u_I - \nabla u_N\|_{L^2(\Omega)} \le C N^{-5/8+\delta} \|f\|_{H^1(\Omega)}, \quad \forall f \in H^1(\Omega). \tag{1.4}$$

Our approach can be easily modified to prove a similar result for average type recovery scheme [47] or polynomial preserving recovery scheme [45]. For example, let us define an average type recovery scheme by $R : \nabla \mathbb{V}_N \mapsto \overline{\mathbb{V}}_N \times \overline{\mathbb{V}}_N$

$$R(\nabla u_N)(x_i) = \frac{\sum_{\tau \in \omega_i} |\tau| \nabla u_N|_\tau}{|\omega_i|}, \text{ for all vertices } x_i \in \mathcal{T},$$

where ω_i is the patch including the vertex x_i, i.e., the union of all triangles containing x_i, and $|\cdot|$ is the two dimensional Lebesgue measure. Then a similar estimate

$$\|\nabla u - R(\nabla u_N)\|_{L^2(\Omega)} \le C N^{-5/8+\delta} \|f\|_{H^1(\Omega)}, \quad \forall f \in H^1(\Omega), \tag{1.5}$$

holds. The average type recovery involves only simple function evaluation and arithmetic operations, and thus is more computationally favorable.

The idea of post-processing the solution in the L^2-norm for a better approximation has been widely addressed. For example, see the early paper [21] in 1974. When the solution u is smooth enough, the superconvergence theory is well estabilished. See [5, 7, 10, 13–15, 27, 29, 36, 38, 46] for the super-closness (1.4); see [7, 15, 22, 28, 30, 41–44] for the superconvergence of recovered gradient (1.3) or (1.5). Analogue of (1.3), (1.4), and (1.5) on quasi-uniform meshes are usually proved with the assumption $u \in H^3(\Omega) \cap W^{2,\infty}(\Omega)$, which is not realistic for corner singularities.

Instead of standard Sobolev spaces, we here use weighted Sobolev spaces to prove similar results on graded meshes for corner singularities, and establish (1.3), (1.4), and (1.5) in terms of the smoothness of f.

It is worth noting that using the knowledge of singular expansions of the solution near the vertices, [23] presented a super-closeness result on smoother graded meshes introduced by [4]. In [34, 35], such expansion is used to prove superconvergence on rectangular meshes. Also in a recent paper [40], the knowledge of singular expansions of the solution near the vertices is used to justify the superconvergence of recovered gradients on adaptive grids obtained from a posteriori processes.

Based on different principles, we use weighted Sobolev spaces to prove the superconvergce of gradient recovery schemes on a class of graded meshes for corner singularities, which can be generated by a simple and explicit process. Since the singular expansion is not required in our analysis, it is possible to extend our results to other singular problems (transmission problems, Schrödinger type operators, and many other singular operators from physics) [31, 33], which can be treated in similar weighted Sobolev spaces.

Throughout this paper, by $x \lesssim y$, we mean $x \leq Cy$, for a generic constant $C > 0$, and by $x \eqsim y$, we mean $x \lesssim y$ and $y \lesssim x$. All constants hidden in this notation are independent of the problem size N and of the solution. However, they may depend on the shape of Ω, and on other parameters which will be specified in the context.

The rest of this paper is organized as follows. In Section 2 we introduce the weighted Sobolev space, the construction of graded meshes, and error estimates on the interpolant and finite element solution. In Section 3, we prove the super-closeness and superconvergence of the recovered gradient. In Section 4, we provide a numerical example to support our theoretical results.

2. Approximation Using Weighted Sobolev Spaces

In this section, we shall briefly introduce the weighted Sobolev space $\mathcal{K}^m_a(\Omega)$, and provide preliminary results in order to carry out further analysis on graded meshes. On details of weighted Sobolev spaces used here, we refer readers to [6, 26, 31]. In addition, we also establish some new error estimates which cannot be found in [6]. Throughout this paper, we assume $\Omega \subset \mathbb{R}^2$ is a polygonal domain with vertices $v_i, i = 1, \cdots, M$. The interior angle at vertex v_i is denoted by α_i for $i = 1, \cdots, M$.

2.1. Weighted Sobolev spaces

Let r_i be the distance function from any point in $\overline{\Omega}$ to the i-th vertex v_i. Denote by l the minimum of non-zero distances from any v_i to an edge of Ω. Let

$$\tilde{l} := \min(1/2, l/4), \qquad \mathcal{V}_i := \Omega \cap B(v_i, \tilde{l}),$$

where $B(x, r)$ denotes the open ball centered at x with radius r. Note that the neighborhoods $\mathcal{V}_i$ of v_i are disjoint for $i = 1, \cdots, M$. We choose a smooth function $\rho \in C^\infty(\Omega) : \overline{\Omega} \to [0, 2\tilde{l}]$ satisfying

$$\rho(x) = r_i, \qquad \text{when } \ x \in \mathcal{V}_i, \quad \text{and}$$
$$\rho(x) \geq \tilde{l}/2, \qquad \text{when } x \in \Omega \backslash \cup \mathcal{V}_i.$$

Such a smooth function can be easily constructed, e.g., using mollifier to smoothly glue r_i in $\mathcal{V}_i$ and the constant function $\tilde{l}/2$ in an open domain inside $\Omega \backslash \cup \mathcal{V}_i$.

This leads to the definition of the weighted Sobolev space for corner singularities [6, 26, 31].

Definition 2.1. *Let ρ be chosen as above, and let $m \in \mathbb{Z}_+$ and $a \in \mathbb{R}$. The weighted Sobolev space $\mathcal{K}_a^m(\Omega)$ is defined as:*

$$\mathcal{K}_a^m(\Omega) := \{v \in L^2(\Omega) : \ \rho^{i+j-a}\partial_x^i \partial_y^j v \in L^2(\Omega), \ \forall \ i+j \leq m, i, j \in \mathbb{N}\}.$$

Equipped with the inner product

$$(u, v)_{\mathcal{K}_a^m(\Omega)} := \sum_{i+j\leq m} \int_\Omega \rho^{2(i+j-a)} (\partial_x^i \partial_y^j u)\, (\partial_x^i \partial_y^j v)\, \mathrm{dxdy},$$

the space $\mathcal{K}_a^m(\Omega)$ is a Hilbert space by the standard argument [20], with the induced norm

$$\|u\|_{\mathcal{K}_a^m(\Omega)} := \left(\sum_{i+j\leq m} \|\rho^{i+j-a}\partial_x^i \partial_y^j u\|^2_{L^2(\Omega)} \right)^{1/2},$$

and semi-norm

$$|u|_{\mathcal{K}_a^m(\Omega)} := \left(\sum_{i+j= m} \|\rho^{i+j-a}\partial_x^i \partial_y^j u\|^2_{L^2(\Omega)} \right)^{1/2}.$$

We here survey some intrinsic properties of the weighted Sobolev space $\mathcal{K}_a^m(\Omega)$ that are necessary for our further analysis. These results can be easily verified by direct calculations. One can also see [6, 26, 31] for details of proofs.

From Definition 2.1, ρ is equal to the distance function in a small neighborhood of the vertex and bounded below away from zero otherwise. Then, it can be seen that $\mathcal{K}_a^m(\Omega)$ and $H^m(\Omega)$ are equivalent on the domain whose closure excludes vertices of Ω.

Lemma 2.1. *For $0 < \xi \leq \tilde{l}/4$, let $\tilde{G} \subset \Omega$ be an open domain, such that $\rho \geq \xi$ on $\tilde{G}$. Then for all $u \in H^m(\tilde{G})$,*

$$M_2 \|u\|_{\mathcal{K}_a^m(\tilde{G})} \leq \|u\|_{H^m(\tilde{G})} \leq M_1 \|u\|_{\mathcal{K}_a^m(\tilde{G})},$$

where M_1 and M_2 depend on ξ, m and a, but not on u.

The following lemma gives relations between different spaces near a vertex of Ω.

Lemma 2.2. *Let $G \subset \mathcal{V}_i$ be an open subset of $\mathcal{V}_i$, such that $\rho \leq \xi \leq \tilde{l}$ on G. Then, for $0 \leq m' \leq m$ and $a' \leq a$, we have*

1. $\mathcal{K}_a^m(\Omega) \subset \mathcal{K}_{a'}^{m'}(\Omega)$;
2. $\|u\|_{\mathcal{K}_{a'}^{m'}(G)} \leq \xi^{a-a'} \|u\|_{\mathcal{K}_a^m(G)}$, $\forall u \in \mathcal{K}_a^m(\Omega)$;
3. $\|u\|_{H^m(G)} \leq \xi^{a-m} \|u\|_{\mathcal{K}_a^m(G)}$, *if* $a \geq m$; *and*
4. $\|u\|_{\mathcal{K}_a^m(G)} \leq \xi^{-a} \|u\|_{H^m(G)}$, *if* $a \leq 0$.

We now give the *homogeneity* argument in the weighted Sobolev space. For simplicity, we consider a new coordinate system that is a simple translation of the old xy-coordinate system with v_i now at the origin of the new coordinate system. Let $G \subset \mathcal{V}_i$ be the subset, such that $\rho \le \xi \le \tilde{l}$ on G. For $0 < \lambda < 1$, we let $G' := \lambda G$ and define the dilation of a function on G in the new coordinate system as follows,

$$v_\lambda(x, y) := v(\lambda x, \lambda y) \tag{2.1}$$

for all $(x, y) \in G \subset \mathcal{V}_i$. The following result can be found at [6] (Lemma 1.9).

Lemma 2.3. *Let $0 < \lambda < 1$ and $G \subset \mathcal{V}_i$ be an open subset such that $G' := \lambda G \subset \mathcal{V}_i$. Then for any $u \in \mathcal{K}^m_a(\mathcal{V}_i)$*

$$\|u_\lambda\|_{\mathcal{K}^m_a(G)} = \lambda^{a-1}\|u\|_{\mathcal{K}^m_a(G')}.$$

In addition, a direct calculation shows that

Lemma 2.4. *Let P be a differential operator of order l, $0 \le l \le 2$. Then*

$$P: \ \mathcal{K}^{m+l}_{1+\epsilon}(\Omega) \cap H^1_0(\Omega) \to \mathcal{K}^m_{1+\epsilon-l}(\Omega),$$

defines a bounded map.

2.2. Regularity of the solution

Given a function $f \in H^{-1}(\Omega) := (H^1_0(\Omega))'$, a weak solution of (1.1) is a function $u \in H^1_0(\Omega)$ satisfying the following weak formulation:

$$a(u, v) = \langle f, v\rangle, \quad \forall\, v \in H^1_0(\Omega), \tag{2.2}$$

where

$$a(u, v) = (\nabla u, \nabla v) = \int_\Omega \nabla u \cdot \nabla v,$$

and $\langle f, v\rangle$ is the dual pair of $H^{-1}(\Omega) \times H^1_0(\Omega)$. In particular, for $f \in L^2(\Omega)$, $\langle \cdot, \cdot\rangle$ can be identified with the L^2-inner product, i.e., $\langle f, v\rangle = (f, v) = \int_\Omega fv$.

Let $\mathcal{T}$ be a triangulation of Ω. We denoted by $\mathbb{V}^m_{\mathcal{T}}$ the finite element space

$$\mathbb{V}^m_{\mathcal{T}} = \{v \in H^1_0(\Omega) : v|_\tau \in \mathcal{P}_m(\tau), \forall \tau \in \mathcal{T}\},$$

where $\mathcal{P}_m(\tau)$ is the polynomial space of order m on the triangle τ.

The finite element approximation of (2.2) is: given a function $f \in H^{-1}(\Omega)$ find $u_{\mathcal{T}} \in \mathbb{V}^m_{\mathcal{T}}$ such that

$$a(u_{\mathcal{T}}, v) = \langle f, v\rangle, \quad \forall\, v \in \mathbb{V}^m_{\mathcal{T}}. \tag{2.3}$$

By the Poincaré inequality, $a(\cdot, \cdot)$ defines an inner product in $H^1_0(\Omega)$, and thus the existence and uniqueness of the solution u and $u_{\mathcal{T}}$ comes from the Riesz representation theorem.

From (2.2) and (2.3), we immediately get the Galerkin orthogonality

$$a(u - u_{\mathcal{T}}, v) = 0, \quad \forall\, v \in \mathbb{V}^m_{\mathcal{T}}.$$

Consequently,

$$\|\nabla(u - u_{\mathcal{T}})\|_{L^2(\Omega)} = \inf_{v \in \mathbb{V}^m_{\mathcal{T}}} \|\nabla(u - v)\|_{L^2(\Omega)}, \tag{2.4}$$

namely $u_{\mathcal{T}}$ is the best approximation of u in the $a(\cdot,\cdot)$ inner product.

When f is more regular, we may expect the solution u to be in high-order Sobolev spaces. Here we recall the regularity result in terms of weighted Sobolev spaces $\mathcal{K}_a^m(\Omega)$ that has been proved in [6,31]. Recall that α_i is the interior angle of the i-th vertex of Ω.

Theorem 2.1. *Let $\beta := \min_i\{\pi/\alpha_i, 1\}$ and $|\epsilon| < \beta$. Then, for any given $f \in \mathcal{K}_{\epsilon-1}^{m-1}(\Omega)$, there exists a unique $u \in \mathcal{K}_{\epsilon+1}^{m+1}(\Omega) \cap H_0^1(\Omega)$ solving the equation (2.2), and*

$$\|u\|_{\mathcal{K}_{\epsilon+1}^{m+1}(\Omega)} \le C(\Omega,\epsilon)\|f\|_{\mathcal{K}_{\epsilon-1}^{m-1}(\Omega)},$$

where C depends on Ω and ϵ, but not on u or f.

We mention that even if the domain is convex, the solution could have singularities near vertices in some Sobolev spaces $H^m(\Omega)$, for $m > 2$. From Theorem 2.1, however, there is no loss of $\mathcal{K}_a^m(\Omega)$-regularity in the weighted Sobolev spaces. Therefore, it is convenient to use weighted Sobolev spaces $\mathcal{K}_a^m$ on non-smooth domains to carry out the analysis.

In particular, for $f \in H^1(\Omega) \subset \mathcal{K}_{\epsilon-1}^1(\Omega)$, $0 < \epsilon < \beta$, the solution $u \in \mathcal{K}_{\epsilon+1}^3(\Omega)$. We will use this property to replace the strong regularity assumption $u \in H^3(\Omega)$ used in the literature, in order to study the superconvergence on graded meshes.

2.3. Graded meshes

Following [6,31], we now construct a class of suitable graded meshes to obtain the optimal convergence rate of the finite element solution in the presence of the corner singularity in the solution of (1.1). Starting from an initial triangulation of Ω, we divide each triangle into four triangles to construct such a sequence of triangulations, which is similar to the regular mid-point refinement. The difference is, in order to attack the corner singularity, when we perform the refinement, we move the middle points of edges towards the singular vertex of Ω. Here a singular vertex v_i means $\alpha_i > \pi$.

Definition 2.2. *Let $\kappa \in (0,1/2]$, and $\mathcal{T}$ be a triangulation of Ω such that each triangle in $\mathcal{T}$ contains at most one vertex of Ω.* The κ-refinement of $\mathcal{T}$, *denoted by $\kappa(\mathcal{T})$, is obtained by dividing each edge AB of $\mathcal{T}$ into two parts as follows. If neither A nor B is a singular vertex of Ω, then we divide AB into two equal parts. Otherwise, if A is a singular vertex of Ω, we divide AB into AC and CB, such that $|AC| = \kappa|AB|$. This will divide each triangle of $\mathcal{T}$ into four triangles.*

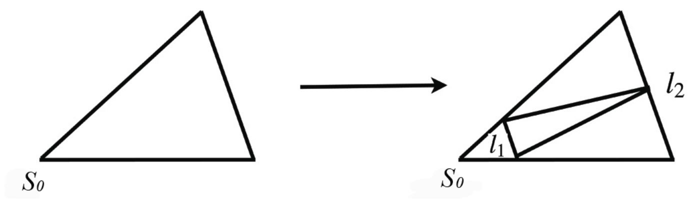


Fig. 2.1. One refinement of triangle T with a singular point S_0, $\kappa = l_1/l_2$

We start with an initial triangulation $\mathcal{T}_0$ of Ω such that all triangles in $\mathcal{T}_0$ have interior angles bounded below by a positive angle and are of comparable sizes. Furthermore each triangle in $\mathcal{T}_0$ contains at most one vertex of Ω. Let $\mathcal{T}_i = \kappa(\mathcal{T}_{i-1})$ for $i = 1, \cdots, L$. We then obtain a sequence of triangulations $\{\mathcal{T}_L\}_{L=1}^{\infty}$ and will use it to construct our finite element space. Note that for a fixed κ, $\{\mathcal{T}_L\}_{L=1}^{\infty}$ is shape regular and the shape regular constant depending only on κ and $\mathcal{T}_0$.

The process of generating triangles in Definition 2.2 actually decomposes the triangulation $\mathcal{T}_L$ into layers $D_i, i = 0, \cdots, L$ that can be described as follows. For $0 \le i < L$, let $\{\tau_{i,j}, j = 1, \cdots K\}$ be the set of the triangles in $\mathcal{T}_i$ that contains the singular vertex S_0. Then, after one refinement, $\tau_{i,j}$ is divided into a small similar triangle with the same vertex S_0 and a trapezoid between two parallel sides. (See Figure 2.1). We thus denote by D_{i+1} the union of the trapezoids generated in $\cup_{j=1}^{K} \tau_{i,j}$ during this refinement. In addition, we define

$$D_L = \cup_{j=1}^{K} \tau_{L,j}, \qquad D_0 = \Omega \backslash \cup_{i=1}^{L} D_i.$$

The generation of different layers D_0 and D_1 is illustrated in Figure 2.2.

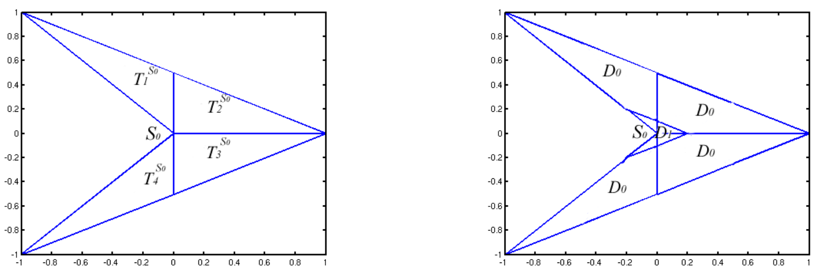


Fig. 2.2. Initial triangles with vertex S_0 on a quadrilateral domain (left); layer D_0 and D_1 after one refinement (right), $\kappa = 0.2$.

From Definition 2.2, we see the layer D_i, $0 < i < L$, is composed of trapezoids, and each trapezoid is divided into three triangles of size $\eqsim \kappa^i$ after the ith-refinement of $\mathcal{T}_0$. Therefore, after L refinements, in the finest triangulation $\mathcal{T}_L$, D_i is decomposed into quasi-uniform triangles of size $h_i \eqsim \kappa^i \, 2^{i-L}$. Note that D_0 consists of all triangles away from singular vertices of Ω, and D_L consists of triangles containing singular vertices with mesh size $h_L = \kappa^L$. We summarize the following important relation of the local mesh size in layers:

$$h_i/\kappa^i \eqsim 2^{i-L}, \quad 0 \le i \le L. \tag{2.5}$$

By the construction, in each layer away from the singular vertex, the weight function ρ is comparable with a constant, i.e.

$$\rho(x) \eqsim \kappa^i, \quad \forall x \in D_i, \; 0 \le i < L. \tag{2.6}$$

Note that (2.6) does not hold for $i = L$, where we can only have $\rho \lesssim \kappa^L$ and cannot bound ρ below by κ^L since ρ is approaching to zero.

Since every triangle is divided into four smaller triangles in one κ-refinement, the number of interior nodes N of $\mathcal{T}_L$ satisfies

$$N \eqsim 4^L. \tag{2.7}$$

The choice of the grading parameter κ, determines the density of the mesh accumulating at the singular vertex, and consequently, determines the approximation property of finite element spaces. It has been shown in [6,31] that if we choose

$$\kappa = 2^{-m/\epsilon}, \tag{2.8}$$

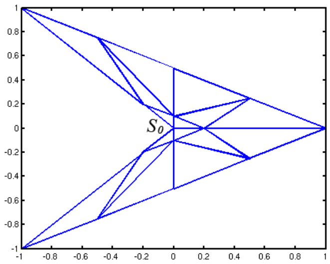


Fig. 2.3. One refinement from the initial triangulation in Figure 2.2, $\kappa = 0.2$

then the quasi-optimal rate of convergence for finite element approximation can be obtained on the graded mesh from the κ-refinement.

All constants skipped in (2.5), (2.6), and (2.7) depend on the initial triangulation and the grading parameter κ. The latter, by (2.8) depends on the choice of m and ϵ. In our applications, we shall usually choose $m = 1$ or 2, and a fixed $\epsilon < \beta$. Here recall that $\beta = \min_i\{\pi/\alpha_i, 1\}$ depends only on the interior angles of Ω. Therefore, all constants depend on the domain Ω and the initial triangulation $\mathcal{T}_0$.

2.4. Interpolation error estimates

In this section, we give the interpolation error estimate in H^1 semi-norm and a weighted L^2 norm on $\mathcal{T}_L$. Let us introduce the standard Lagrange interpolation operator $I_m : H_0^1(\Omega) \cap C(\bar{\Omega}) \to \mathbb{V}_{\mathcal{T}_L}^m$. In particular, we use the short notation $u_I = I_1 u$ and $u_{II} = I_2 u$ for the linear and quadratic interpolant, respectively.

Lemma 2.5. *Let $\tau_i \in \mathcal{T}_L$ be a triangle sitting in the layer D_i, $0 \le i \le L$. Then for all $u \in \mathcal{K}_{\epsilon+1}^{m+1}(\Omega), m \ge 1, \epsilon > 0$, there exists a constant $C = C(\kappa, m, \mathcal{T}_0)$ such that*

$$\|\nabla(u - I_m u)\|_{L^2(\tau_i)} \le C\, \kappa^{i\epsilon}(h_i/\kappa^i)^m \|u\|_{\mathcal{K}_{\epsilon+1}^{m+1}(\tau_i)}, \quad \text{and} \tag{2.9}$$

$$\|u - I_m u\|_{\mathcal{K}_1^0(\tau_i)} \le C\, \kappa^{i\epsilon}(h_i/\kappa^i)^{m+1} \|u\|_{\mathcal{K}_{\epsilon+1}^2(\tau_i)}. \tag{2.10}$$

Proof. Recall that we have the standard interpolation error estimate

$$\|\nabla(u - I_m u)\|_{L^2(\tau_i)} \le C h_i^m |u|_{H^{m+1}(\tau_i)}.$$

On D_i, $0 \le i < L$, by (2.6),

$$|u|_{H^{m+1}(\tau_i)} \le C(\kappa) 1/(\kappa^i)^{m-\epsilon} |u|_{\mathcal{K}_{\epsilon+1}^{m+1}(\tau_i)},$$

which leads to

$$\|\nabla(u - I_m u)\|_{L^2(\tau_i)} \le C(\kappa)\kappa^{i\epsilon}(h_i/\kappa^i)^m |u|_{\mathcal{K}_{\epsilon+1}^{m+1}(\tau_i)}.$$

The estimate (2.10) for $0 \le i < L$ is proved similarly.

For the most inner layer, i.e., $i = L$, we need a special treatment. Let τ_L be a triangle in D_L. We denote $u_\lambda(x, y) = u(\lambda x, \lambda y)$ with the single singular vertex v_i as the origin. Let $\lambda = C\kappa^L$ such that $\hat{\tau}_L := \tau_L/\lambda$ is in $\mathcal{V}_i$. Then, $u_\lambda(x, y) \in \mathcal{K}_{\epsilon+1}^{m+1}(\hat{\tau}_L)$ by Lemma 2.3.

Let $\chi : \hat{\tau}_L \to [0,1]$ be a non-decreasing smooth function that depends only on ρ and is equal to 0 in a neighborhood of S_0, but is equal to 1 at all other interpolant points of $\hat{\tau}$ different from S_0. We introduce the auxiliary function $v = \chi u_\lambda$ on $\hat{\tau}_L$. Consequently,

$$|v|_{\mathcal{K}_1^{m+1}(\hat{\tau}_L)} = |\chi u_\lambda|_{\mathcal{K}_1^{m+1}(\hat{\tau}_L)} \leq C|u_\lambda|_{\mathcal{K}_1^{m+1}(\hat{\tau}_L)},$$

where C depends on the choice of the nodal points. In addition, since we chop out the singular region, $v \in H^{m+1}(\hat{\tau}_L)$, and

$$\|\nabla(v - I_m v)\|_{L^2(\tau_L)} = |v - I_m v|_{\mathcal{K}_1^1(\hat{\tau}_L)} \lesssim |v|_{\mathcal{K}_1^{m+1}(\hat{\tau}_L)}.$$

Since we are considering the homogenous Dirichlet boundary condition, $u(S_0) = 0$ and the interpolant $I_m v = I_m u_\lambda = (I_m u)_\lambda$ by the definition of v and the affine invariant of the Lagrange interpolatoin.

By the homogeneity argument (see Lemma 2.3), we have

$$\begin{aligned}
|u - I_m u|_{\mathcal{K}_1^1(\tau_L)} &= |u_\lambda - I_m u_\lambda|_{\mathcal{K}_1^1(\hat{\tau}_L)} \\
&\leq |u_\lambda - v|_{\mathcal{K}_1^1(\hat{\tau}_L)} + |v - I_m u_\lambda|_{\mathcal{K}_1^1(\hat{\tau}_L)} \\
&= |u_\lambda - v|_{\mathcal{K}_1^1(\hat{\tau}_L)} + |v - I_m v|_{\mathcal{K}_1^1(\hat{\tau}_L)} \\
&\lesssim |u_\lambda|_{\mathcal{K}_1^1(\hat{\tau}_L)} + \|v\|_{\mathcal{K}_1^{m+1}(\hat{\tau}_L)} \\
&\lesssim |u_\lambda|_{\mathcal{K}_1^1(\hat{\tau}_L)} + \|u_\lambda\|_{\mathcal{K}_1^{m+1}(\hat{\tau}_L)} \\
&\lesssim |u|_{\mathcal{K}_1^1(\tau_L)} + \|u\|_{\mathcal{K}_1^{m+1}(\tau_L)} \\
&\lesssim \kappa^{L\epsilon} \|u\|_{\mathcal{K}_{\epsilon+1}^{m+1}(\tau_L)}.
\end{aligned}$$

The first and the sixth relations above are due to Lemma 2.3; and the seventh is based on Lemma 2.2. Since $|\cdot|_{\mathcal{K}_1^1(\tau_i)} \eqsim |\cdot|_{H^1(\tau_i)}$ by the definition, we complete the proof for the H^1-estimate on the triangle.

The $\mathcal{K}_1^0$-estimate is based on a similar calculation on $\hat{\tau}_L$ and Lemma 2.3. □

The global estimates on the H^1-error or the $\mathcal{K}_1^1$ -error then follow from summing up this result on all triangles. As a consequence of (2.9), we obtain the optimal convergence rate of nodal interpolation in H^1-norm. Here we only present results for linear interpolation u_I, i.e., $m = 1$.

Theorem 2.2. *Suppose the grading parameter $\kappa = 2^{-1/\epsilon}$, $0 < \epsilon < \beta$ and $u \in \mathcal{K}_{1+\epsilon}^2(\Omega)$. Then there exists a constant $C = C(\epsilon, \mathcal{T}_0)$*

$$\|\nabla(u - u_I)\|_{L^2(\Omega)} \leq C\, N^{-1/2} \|u\|_{\mathcal{K}_{1+\epsilon}^2(\Omega)}.$$

Proof. From the relation of h_i, κ_i and N; see (2.5), (2.6), (2.7), and (2.8), we have

$$\|\nabla(u - u_I)\|_{L^2(\tau_i)} \leq C(\epsilon, \mathcal{T}_0)\, N^{-1/2} \|u\|_{\mathcal{K}_{\epsilon+1}^2(\tau_i)},$$

which leads to the desired result by standard summation process. □

We now give an estimate of $u - u_I$ in a weighted L^2 norm. We first define a piecewise constant approximation of the weight function $\rho^{\epsilon-1}$:

$$r_c|_{D_i} = (2\kappa)^{-i}, i = 0, \cdots, L.$$

Then, we define the weighted L^2 inner product with respect to r_c^2,

$$(u,v)_{r_c^2} = (r_c\, u, r_c\, v) = \int_\Omega r_c^2\, uv, \tag{2.11}$$

In addition, the above inner product induces the norm,

$$\|u\|_{r_c, L^2(\Omega)} = (u,u)_{r_c^2}^{1/2} = \|r_c\, u\|_{L^2(\Omega)},$$

Recall $\rho \lesssim \kappa^i$ in D_i, $0 \le i \le L$. Thus $r_c|_{D_i} \lesssim \rho^{\epsilon-1}$ and $\|u\|_{r_c,L^2(\Omega)}$ can be thought as an approximation of the weighted Sobolev norm $\|u\|_{\mathcal{K}^0_{\epsilon-1}(\Omega)}$.

Theorem 2.3. *Suppose the grading parameter $\kappa = 2^{-1/\epsilon}, \epsilon > 0$ and $u \in \mathcal{K}^2_{1+\epsilon}(\Omega)$. Then there exists a constant $C = C(\epsilon, \mathcal{T}_0)$*

$$\|u - u_I\|_{r_c, L^2(\Omega)} \le C\, N^{-1} \|u\|_{\mathcal{K}^2_{1+\epsilon}(\Omega)}.$$

Proof. By the definition of $(\cdot,\cdot)_{r_c, L^2(\Omega)}$ and the weighted Sobolev space $\mathcal{K}^0_1(\Omega)$,

$$\|u - u_I\|^2_{r_c,L^2(\Omega)} = \sum_{i=0}^{L} \|2^{-i}\kappa^{-i}(u-u_I)\|^2_{L^2(D_i)} \lesssim \sum_{i=0}^{L} 2^{-2i} \|u - u_I\|^2_{\mathcal{K}^0_1(D_i)}.$$

Now using the estimate (2.10), we get

$$\|u - u_I\|^2_{r_c,L^2(\Omega)} \lesssim N^{-2} \sum_{i=0}^{L} \|u\|^2_{\mathcal{K}^2_{1+\epsilon}(D_i)} = N^{-2} \|u\|^2_{\mathcal{K}^2_{1+\epsilon}(\Omega)}.$$ □

Note that $\kappa \le 1/2$ and thus the weight $r_c \ge 1$. As a consequence

$$\|u - u_I\|_{L^2(\Omega)} \le \|u - u_I\|_{r_c,L^2(\Omega)} \le C(\kappa) N^{-1} \|u\|_{\mathcal{K}^2_{1+\epsilon}(\Omega)}.$$

We can prove a more tight estimate for the standard L^2-norm provided with a stronger grading parameter.

Theorem 2.4. *Suppose the grading parameter $\kappa = 2^{-2/\epsilon}, \epsilon > 0$ and $u \in \mathcal{K}^2_\epsilon(\Omega)$. Then there exists a constant $C = C(\epsilon, \mathcal{T}_0)$*

$$\|u - u_I\|_{L^2(\Omega)} \le C\, N^{-1} \|u\|_{\mathcal{K}^2_\epsilon(\Omega)}.$$

Proof. Let us take $\tau_i \in D_i$ for $0 \le i < L$. By the standard interpolation error estimate and the relation of ρ and κ,

$$\|u - u_I\|_{L^2(\tau_i)} \lesssim h_i^2 |u|_{H^2(\tau_i)} \le \kappa^{i\epsilon} (h_i/\kappa^i)^2 |u|_{\kappa^2_\epsilon(\tau_i)} \lesssim N^{-1} \|u\|_{\kappa^2_\epsilon(\tau_i)}.$$

In the last step, the choice of $\kappa = 2^{-2/\epsilon}$ is important to get the correct rate.

The case $i = L$ is proved using a similar technique in Lemma 2.5. □

2.5. Error estimate on the finite element approximation

In this section, we shall give the error estimate on $u - u_N$ in the H^1 norm and the weighted L^2 norm. Throughout this section, we always assume u is the solution of (2.2), $u_N \in \mathbb{V}_L := \mathbb{V}^1_{\mathcal{T}_L}$ is the linear finite element approximation of u, i.e., the solution of (2.3), and the data $f \in L^2(\Omega)$. Note that we use the subscript N in u_N to indicate the relation of the finite element solution with the number of interior nodes which is also the dimension of $\mathbb{V}_L$.

Theorem 2.5. *Suppose the grading parameter κ satisfies $\kappa = 2^{-1/\epsilon}$, for $0 < \epsilon < \beta$. There exists a constant depending only on $\epsilon, \mathcal{T}_0$ and Ω, such that,*

$$\|\nabla(u - u_N)\|_{L^2(\Omega)} \le CN^{-1/2}\|f\|_{L^2(\Omega)}.$$

Proof. By (2.4) and Theorem 2.2,

$$\|\nabla(u - u_N)\|_{L^2(\Omega)} \le \|\nabla(u - u_I)\|_{L^2(\Omega)} \lesssim N^{-1/2}\|u\|_{\mathcal{K}^2_{1+\epsilon}(\Omega)}.$$

Then by the regularity result (Theorem 2.1) and Lemma 2.2,

$$\|u\|_{\mathcal{K}^2_{1+\epsilon}(\Omega)} \lesssim \|f\|_{\mathcal{K}^0_{-1+\epsilon}(\Omega)} \lesssim \|f\|_{L^2(\Omega)}. \qquad \square$$

We now give a weighted L^2 error estimate of $u - u_N$ using the standard duality argument. Similar estimates are obtained in [4, 11].

Theorem 2.6. *Suppose the grading parameter κ satisfies $\kappa = 2^{-1/\epsilon}$, for $0 < \epsilon < \beta$. There exists a constant depending only on $\epsilon, \mathcal{T}_0$ and Ω, such that*

$$\|u - u_N\|_{r_c, L^2(\Omega)} \le C\, N^{-1}\|f\|_{L^2(\Omega)}.$$

Proof. Consider the following boundary value problem: Find $w \in H^1_0(\Omega)$ such that

$$a(w, v) = (u - u_N, v)_{r_c^2} = (r_c^2(u - u_N), v) \quad \forall v \in H^1_0(\Omega). \tag{2.12}$$

Then by choosing $v = u - u_N$ in (2.12), we have

$$\begin{aligned} \|u - u_N\|^2_{r_c, L^2(\Omega)} &= (u - u_N, u - u_N)_{r_c^2} \\ &= a(w, u - u_N) = a(w - w_I, u - u_N) \\ &\le \|\nabla(w - w_I)\|\|\nabla(u - u_N)\|. \end{aligned}$$

By Theorems 2.2 and 2.1, we have

$$\|\nabla(w - w_I)\| \le CN^{-1/2}\|w\|_{\mathcal{K}^2_{1+\epsilon}(\Omega)} \le CN^{-1/2}\|r_c^2(u - u_N)\|_{\mathcal{K}^0_{-1+\epsilon}(\Omega)}.$$

By the relation of ρ and r_c in D_i, we have

$$\begin{aligned} \|r_c^2(u - u_N)\|^2_{\mathcal{K}^0_{-1+\epsilon}(\Omega)} &= \sum_{i=0}^{L} \|\rho^{1-\epsilon} r_c^2(u - u_N)\|^2_{L^2(D_i)} \\ &\lesssim \sum_{i=0}^{L} \|r_c(u - u_N)\|^2_{L^2(D_i)} = \|u - u_N\|^2_{r_c, L^2(\Omega)}. \end{aligned}$$

Combining the results above, we have

$$\|u - u_N\|^2_{r_c, L^2(\Omega)} \lesssim N^{-1/2}\|u - u_N\|_{r_c, L^2(\Omega)}\|\nabla(u - u_N)\|_{L^2(\Omega)},$$

which implies

$$\|u - u_N\|_{r_c, L^2(\Omega)} \lesssim N^{-1/2}\|\nabla(u - u_N)\|_{L^2(\Omega)} \lesssim N^{-1}\|f\|_{L^2(\Omega)}. \qquad \square$$

3. Superconvergence on Graded Mesh

In this section, we shall prove the super-closeness between u_N and u_I on graded meshes, on which we have the optimal rate of convergence for quadratic elements. Namely, we choose $\kappa = 2^{-2/\epsilon}$ for the mesh grading. Based on this result, we prove the L^2 projection of ∇u_N will give a better approximation of ∇u. We also sketch the proof for the average type gradient recovery scheme.

3.1. Super-closeness

For a given node x_i, we denote by ω_i, the patch of x_i, which is the union of all triangles sharing this node. The superconvergence comes from the symmetry of the local patch ω_i. Note that to obtain a better convergence rate, we need a graded mesh with higher mesh density at the singular vertices ($\kappa = 2^{-2/\epsilon}$).

Lemma 3.1. *Suppose the grading parameter κ satisfies $\kappa = 2^{-2/\epsilon}$, for some $\epsilon \in (0, \beta)$. For an interior node x_i and $\omega_i \subset D_i, 0 \le i < L$, if $u \in \mathcal{K}^3_{1+\epsilon}(\omega_i)$ and the patch ω_i is symmetric, then we have*

$$a(u - u_I, \varphi_i) \lesssim N^{-1}\|u\|_{\mathcal{K}^3_{1+\epsilon}(\omega_i)}$$

Proof. By the construction of graded meshes, if the patch is symmetric, then it contains six similar triangles. Note that the tangential derivative of φ_i vanishes on the boundary edges of ω_i while for the interior edges of ω_i, the two triangles sharing that edge forms a parallelogram. Using the identity of error formula in [7] (see also [17]), we obtain

$$a(u - u_I, \varphi_i) \lesssim h_i^2\|u\|_{H^3(\omega_i)}\|\nabla\varphi_i\|_{L^2(\omega_i)}.$$

Noting that $\|\nabla\varphi_i\|_{L^2(\omega_i)} \le C$ and using the relation of ρ, κ and h_i, we get

$$h_i^2\|u\|_{3,\omega_i} \lesssim \kappa^{i\epsilon}(h_i/\kappa_i)^2\|u\|_{\mathcal{K}^3_{1+\epsilon}(\omega_i)} \lesssim N^{-1}\|u\|_{\mathcal{K}^3_{1+\epsilon}(\omega_i)}. \qquad \square$$

We then estimate the consistence error on the patch which is not symmetric.

Lemma 3.2. *Suppose the grading parameter κ satisfies $\kappa = 2^{-2/\epsilon}$, for some $\epsilon \in (0, \beta)$. For a node x_i with $\omega_i \subset D_i, 0 \le i \le L$, if $u \in \mathcal{K}^2_{1+\epsilon}(\omega_i)$, we have*

$$a(u - u_I, \varphi_i) \lesssim 2^{L-i}N^{-1}\|u\|_{\mathcal{K}^2_{1+\epsilon}(\omega_i)}. \tag{3.1}$$

Proof. By the Cauchy-Schwarz inequality and interpolation error estimate, we have

$$a(u - u_I, \varphi_i) \le \|\nabla(u - u_I)\|_{L^2(\omega_i)}\|\nabla\phi\|_{L^2(\omega_i)} \lesssim \kappa^{i\epsilon}h_i/\kappa^i\|u\|_{\mathcal{K}^2_{1+\epsilon}(\omega_i)}.$$

By the relation of h_i and κ_i,

$$\kappa^{i\epsilon}h_i/\kappa^i = 2^{-2i}2^{i-L} = 2^{L-i}4^{-L} = 2^{L-i}N^{-1}. \qquad \square$$

When i is close to zero, e.g., $i = 0$, the rate in (3.1) becomes $2^L N^{-1} = N^{-1/2}$. But when $i = L, L-1$, the rate becomes N^{-1}. This estimate indicates when the patch is close to the singularity, although we can only have first order convergence in terms of h_i, the patch is small enough to obtain a better rate N^{-1} in terms of N.

Let $e_i = (u_N - u_I)(x_i)$ and $\boldsymbol{e} = (e_1, \cdots, e_N)^t$. We now estimate $\|\boldsymbol{e}\| := \|\boldsymbol{e}\|_{l^2}$.

Lemma 3.3. *Suppose the grading parameter κ satisfies $\kappa = 2^{-1/\epsilon}$, for some $\epsilon \in (0,\beta)$ and $u \in \mathcal{K}^2_{1+\epsilon}(\Omega)$, $\epsilon > 0$. Then*

$$\|\boldsymbol{e}\| \lesssim N^{-1/2}\|u\|_{\mathcal{K}^2_{1+\epsilon}(\Omega)}.$$

Proof. By the definition of $\boldsymbol{e}$, we have

$$(r_c\sum_i e_i\varphi_i, r_c\sum_j e_j\varphi_j) = \|u_N - u_I\|^2_{r_c,L^2(\Omega)} = \boldsymbol{e}^t\boldsymbol{M}\boldsymbol{e},$$

where r_c is defined in (2.11) and $\boldsymbol{M} = (m_{i,j})$ with

$$m_{i,j} = (\varphi_i, \varphi_j)_{r_c^2} \eqsim r_c^2 h_i^2 = (2\kappa)^{-2i}\kappa^{2i}2^{2i-2L} \eqsim N^{-1}.$$

Therefore, as a symmetric and positive definite and diagonal dominant matrix, $N^{-1} \lesssim \lambda_{\min}(\boldsymbol{M}) \leq \lambda_{\max}(\boldsymbol{M}) \lesssim N^{-1}$ by its definition and the inverse inequality. Then, we conclude

$$\begin{aligned}\|\boldsymbol{e}\| &\leq CN^{1/2}||u_N - u_I||_{r_c,L^2(\Omega)}\\ &\leq CN^{1/2}(||u - u_N||_{r_c,L^2(\Omega)} + ||u - u_I||_{r_c,L^2(\Omega)})\\ &\leq CN^{-1/2}\|u\|_{\mathcal{K}^2_{1+\epsilon}(\Omega)}.\end{aligned}$$

In the last step, we have used the estimates in Theorems 2.3 and 2.6. □

Now it is in the position to prove our main result.

Theorem 3.1. *Suppose the grading parameter κ satisfies $\kappa = 2^{-2/\epsilon}$, for some $\epsilon \in (0,\beta)$, and $f \in H^1(\Omega)$. For any $\delta > 0$, there exists a constant $C = C(\delta, \epsilon, \beta, \mathcal{T}_0)$ such that*

$$\|\nabla u_N - \nabla u_I\| \leq C\, N^{-5/8+\delta}\|f\|_{H^1(\Omega)}.$$

Proof. Let $r_i = a(u - u_I, \varphi_i)$ and $\boldsymbol{r} = (r_1, \cdots, r_N)^t$. Then

$$\begin{aligned}\|\nabla u_N - \nabla u_I\|^2 &= a(u_N - u_I, u_N - u_I)\\ &= a(u - u_I, \sum_{i=1}^N e_i\varphi_i)\\ &= \sum_{i=1}^N r_i e_i \leq \|\boldsymbol{r}\|\|\boldsymbol{e}\|.\end{aligned}$$

We define $\mathcal{I}_{\rm good} = \{1 \leq k \leq N, \omega_k \text{ is symmetric}\}$ and $\mathcal{I}_i = \{1 \leq k \leq N, k \notin \mathcal{I}_{\rm good} \text{ and } \omega_k \subset D_i \cup D_{i-1}\}$. Note that $\mathcal{I}_i$ contains the patch on the boundary of D_i which forms a narrow strip with measure

$$\sum_{k\in\mathcal{I}_i} |\omega_k| = C\kappa^i h_i. \tag{3.2}$$

To estimate the error $\|\boldsymbol{r}\|$, we divide it into two parts

$$\sum_{k=1}^N r_k^2 = \sum_{k\in\mathcal{I}_{\rm good}} r_k^2 + \sum_{i=0}^L\sum_{k\in\mathcal{I}_i} r_k^2 = I_1 + I_2.$$

For $k \in \mathcal{I}_{\rm good}$, note that if the patch is symmetric, then it sits in the interior of some $D_i, i < L$, and thus we can apply Lemma 3.1 to obtain

$$\sum_{i\in\mathcal{I}_{\rm good}} r_k^2 \lesssim N^{-2}\sum_{i\in\mathcal{I}_{\rm good}} \|u\|^2_{\mathcal{K}^3_{1+\epsilon}(\omega_i)} \lesssim N^{-2}\|u\|^2_{\mathcal{K}^3_{1+\epsilon}(\Omega)}.$$

We then estimate the second term I_2. Again, we divide it into two cases, $0 \le i < L$ and $i = L$. When $i \in \mathcal{I}_L$, we apply Lemma 3.2 for $i = L$ or $L-1$ to obtain

$$\sum_{k \in I_L} r_k^2 \lesssim N^{-2} \|u\|^2_{\mathcal{K}^2_{1+\epsilon}(\Omega)}.$$

For $k \in \mathcal{I}_i, 0 \le i < L$, we use the infinity norm estimate. For any $0 < s < 1$,

$$\begin{aligned} a(u - u_I, \varphi_k) &\le \kappa^{i(\epsilon-2)} \|\kappa^{i(2-\epsilon)}(\nabla u - \nabla u_I)\|_{\infty,\omega_k} \|\nabla \varphi_k\|_{L^1(\omega_k)} \\ &\le h_i^s \kappa^{i(\epsilon-2)} |\kappa^{i(2-\epsilon)} u|_{W^{1+s,\infty}(\omega_k)} |\omega_k|^{1/2} \\ &\le C h_i^s \kappa^{i(\epsilon-2)} |\rho^{(2-\epsilon)} u|_{W^{1+s,\infty}(\omega_k)} |\omega_k|^{1/2}. \end{aligned}$$

Note

$$|\rho^{2-\epsilon} u|_{W^{l,\infty}(\omega_k)} \ge C |\kappa^{i(2-\epsilon)} u|_{W^{l,\infty}(\omega_k)}, \quad l = 1, 2.$$

Then, the last inequality above is based on the fact that $W^{1+s,\infty}(\omega_k)$ is defined by interpolation between $W^{1,\infty}(\omega_k)$ and $W^{2,\infty}(\omega_k)$, and therefore, its norm only depends on the corresponding norms of the original spaces [1, 8, 19]. And thus

$$\begin{aligned} \left(\sum_{k \in \mathcal{I}_i} r_k^2 \right)^{1/2} &\lesssim h_i^s \kappa^{i(\epsilon-2)} |\rho^{(2-\epsilon)} u|_{W^{1+s,\infty}(\omega_k)} \left(\sum_{k \in \mathcal{I}_i} |\omega_k| \right)^{1/2} \\ &\lesssim h_i^{s+1/2} \kappa^{i/2} \kappa^{i(\epsilon-2)} |\rho^{(2-\epsilon)} u|_{W^{1+s,\infty}(\Omega \backslash D_L)}. \end{aligned}$$

Here we use the fact (3.2). The rate is computed as the follows

$$\begin{aligned} h_i^{s+1/2} \kappa^{i/2} \kappa^{i(\epsilon-2)} &= \kappa^{i\epsilon} (h_i/\kappa^i)^{s+1/2} \kappa^{i(s-1)} \\ &= 2^{-2i} 2^{(1/2+s)(i-L)} 2^{2i(1-s)/\epsilon} \\ &\eqsim 2^{-i/2} N^{-3/4+\delta}, \end{aligned}$$

where $\delta > 0$ can be arbitrarily small as $s \to 1$. Therefore,

$$\begin{aligned} \sum_{i=0}^{L-1} \sum_{k \in \mathcal{I}_i} r_k^2 &\lesssim N^{-3/2+2\delta} |\rho^{(2-\epsilon)} u|^2_{W^{1+s,\infty}(\Omega \backslash D_L)} \sum_{i=0}^{L-1} 2^{-i/2} \\ &\le C N^{-3/2+2\delta} |\rho^{(2-\epsilon)} u|^2_{W^{1+s,\infty}(\Omega \backslash D_L)}. \end{aligned}$$

Recall $\rho \eqsim \kappa^i$ on $D_i, i < L$. Then, it can been seen that

$$\|\rho^{(2-\epsilon)} u\|_{H^3(\Omega \backslash D_L)} \le C \|u\|_{\mathcal{K}^3_{1+\epsilon}(\Omega \backslash D_L)} \le C \|u\|_{\mathcal{K}^3_{1+\epsilon}(\Omega)}.$$

Therefore, by the Sobolev imbedding theorem, for any $0 < s < 1$,

$$\|\rho^{(2-\epsilon)} u\|_{W^{1+s,\infty}(\Omega \backslash D_L)} \le C \|\rho^{(2-\epsilon)} u\|_{H^3(\Omega \backslash D_L)} \le C \|u\|_{\mathcal{K}^3_{1+\epsilon}(\Omega)}, \tag{3.3}$$

where C depends only on the domain $\Omega \backslash D_L$ and s.

From Lemma 3.3,

$$\|\boldsymbol{e}\| \lesssim N^{1/2} (\|u - u_I\|_{r_c} + \|u - u_N\|_{r_c}) \lesssim N^{-1/2} \|u\|_{\kappa^2_{1+\epsilon}(\Omega)}.$$

Here we apply Lemma 3.3 for $\kappa = 2^{-1/\epsilon'}$ with $\epsilon' = \epsilon/2$ and the fact $\|u\|_{\mathcal{K}^2_{1+\epsilon/2}(\Omega)} \le \|u\|_{\mathcal{K}^2_{1+\epsilon}(\Omega)}$.

Put all estimates together, we obtain

$$\|\nabla u_I - \nabla u_N\|_{L^2(\Omega)} \le \left(\|\boldsymbol{r}\| \|\boldsymbol{e}\| \right)^{1/2} \lesssim N^{-5/8+\delta} \|u\|_{\kappa^3_{1+\epsilon}(\Omega)},$$

which leads to the desired result by the regularity result. □

3.2. Superconvergence of the recovered gradient

In this subsection, we aim to estimating $\|\nabla u - Q_N \nabla u_N\|$. Here recall that Q_N denotes the L^2 projection to $\overline{\mathbb{V}}_N$ and

$$\overline{\mathbb{V}}_N = \{v \in H^1(\Omega) : v|_\tau \in \mathcal{P}_1(\tau), \forall \tau \in \mathcal{T}\},$$

is the linear finite element spaces including the boundary nodes also. Since $\bar{N} := \dim(\overline{\mathbb{V}}_N) \leq CN$, we shall still use N, the number of interior nodes, in the following estimates.

Following [7], we apply the triangle inequality

$$\begin{aligned}
&\|\nabla u - Q_N \nabla u_N\|_{L^2(\Omega)} \\
\leq &\|\nabla u - Q_N \nabla u\|_{L^2(\Omega)} + \|Q_N(\nabla u - \nabla u_I)\|_{L^2(\Omega)} + \|Q_N(\nabla u_I - \nabla u_N)\|_{L^2(\Omega)} \\
= &I_1 + I_2 + I_3,
\end{aligned}$$

and estimate these three terms one by one. Similar approach may be found at [2].

Lemma 3.4. *Suppose the grading parameter $\kappa = 2^{-2/\epsilon}$ for some $\epsilon \in (0, \beta)$ and $u \in \mathcal{K}^3_{1+\epsilon}(\Omega)$. Then*

$$\|\nabla u - Q_N \nabla u\|_{L^2(\Omega)} \leq CN^{-1} \|u\|_{\mathcal{K}^3_{1+\epsilon}(\Omega)}.$$

Proof. We use Theorem 2.4 to estimate I_1 as

$$\|\nabla u - Q_N \nabla u\|_{L^2(\Omega)} \leq \|\nabla u - (\nabla u)_I\|_{L^2(\Omega)} \lesssim N^{-1} \|\nabla u\|_{\mathcal{K}^2_\epsilon(\Omega)}.$$

Since $\nabla : \mathcal{K}^3_{1+\epsilon}(\Omega) \cap H^1_0(\Omega) \to \mathcal{K}^2_\epsilon(\Omega)$ is a bounded operator from Lemma 2.4, i.e.

$$\|\nabla u\|_{\mathcal{K}^2_\epsilon(\Omega)} \leq C \|u\|_{\mathcal{K}^3_{1+\epsilon}(\Omega)},$$

we finish the proof. □

The third term I_3 is from the super-closeness and the stability of Q_N in L^2 norm.

Lemma 3.5. *Suppose the grading parameter $\kappa = 2^{-2/\epsilon}$ for some $\epsilon \in (0, \beta)$ and $u \in \mathcal{K}^3_{1+\epsilon}(\Omega)$. Then for any $\delta > 0$,*

$$\|Q_N(\nabla u_I - \nabla u_N)\|_{L^2(\Omega)} \leq \|\nabla u_I - \nabla u_N\|_{L^2(\Omega)} \lesssim N^{-5/8+\delta} \|u\|_{\mathcal{K}^3_{1+\epsilon}(\Omega)}.$$

So we only need to estimate the second term I_2.

Lemma 3.6. *Suppose the grading parameter $\kappa = 2^{-2/\epsilon}$ for some $\epsilon \in (0, \beta)$ and $u \in \mathcal{K}^3_{1+\epsilon}(\Omega)$. Then for any $\delta > 0$,*

$$\|Q_N(\nabla u - \nabla u_I)\|_{L^2(\Omega)} \lesssim N^{-5/8+\delta} \|u\|_{\mathcal{K}^3_{1+\epsilon}(\Omega)}$$

Proof. First

$$\|Q_N(\nabla u - \nabla u_I)\|_{L^2(\Omega)} \leq \|Q_N(\partial_x u - \partial_x u_I)\|_{L^2(\Omega)} + \|Q_N(\partial_y u - \partial_y u_I)\|_{L^2(\Omega)}.$$

Without loss of generality, we only estimate $\|Q_N(\partial_x u - \partial_x u_I)\|_{L^2(\Omega)}$.

Let $\boldsymbol{s} = (s_1, \cdots, s_{\bar{N}}), s_i = (\partial_x u - \partial_x u_I, \varphi_i)$ and $\overline{\boldsymbol{M}} = (m_{i,j}), m_{i,j} = (\varphi_i, \varphi_j) \eqsim h_i^2$ be the mass matrix. Then by the definition of Q_N,

$$Q_N(\partial_x u - \partial_x u_I) = (\varphi_1, \cdots, \varphi_N) \cdot \overline{\boldsymbol{M}}^{-1} \boldsymbol{s}$$

and thus

$$\|Q_N(\partial_x u - \partial_x u_I)\|^2 = \boldsymbol{s}^t \overline{\boldsymbol{M}}^{-1} \overline{\boldsymbol{M}\boldsymbol{M}}^{-1} \boldsymbol{s} = \boldsymbol{s}^t \overline{\boldsymbol{M}}^{-1} \boldsymbol{s} \le C \sum_{i=1}^{\bar{N}} h_i^{-2} s_i^2.$$

We shall estimate $h_i^{-1} s_i$ as before. For interior nodes and $\omega_i \in D_i, i < L$ is symmetric, by integration by parts, we have

$$s_i = (\partial_x u - \partial_x u_I, \varphi_i) = (u - u_I, \partial_x \varphi_i).$$

When the patch ω_i is symmetric, for any quadratic function p, $(p - p_I, \partial_x \varphi_i) = 0$, since $p - p_I$ is even and $\partial_x \varphi_i$ is odd (with respect to the node x_i). We shall also use the quadratic interpolant u_Π as a bridge in the proof. It is obvious that $u_I = (u_\Pi)_I$.

Let $p \in \mathcal{P}_2(\omega_i)$ be any quadratic polynomial in ω_i. Then

$$\begin{aligned}(u - u_I, \partial_x \varphi_i) &= (u - p + p_I - u_I, \partial_x \varphi_i)\\ &\le \left(\|u - p\|_{L^2(\omega_i)} + \|(p - u_\Pi)_I\|_{L^2(\omega_i)}\right) \|\nabla \varphi_i\|_{L^2(\omega_i)}\\ &\le C\|u - p\|_{L^2(\omega_i)} + \|p - u_\Pi\|_{L^2(\omega_i)}\\ &\le C\|u - p\|_{L^2(\omega_i)} + \|u - u_\Pi\|_{L^2(\omega_i)}.\end{aligned}$$

By the interpolation error estimate

$$\|u - u_\Pi\|_{L^2(\omega_i)} \lesssim h_i^3 |u|_{H^3(\omega_i)}.$$

Since $p \in \mathcal{P}_2(\omega_i)$ is arbitrary, we can choose p^*, the L^2 projection of u into $\mathcal{P}_2(\omega_i)$, and use the Bramble-Hilbert lemma to get

$$\|u - p^*\|_{L^2(\omega_i)} = \inf_{p \in \mathcal{P}_2(\omega_i)} \|u - p\|_{L^2(\omega_i)} \lesssim h_i^3 |u|_{H^3(\omega_i)}.$$

Therefore

$$h_i^{-1} s_i \lesssim h_i^2 |u|_{H^3(\omega_i)} \lesssim (h_i/\kappa^i)^2 \kappa^{i\epsilon} |u|_{\mathcal{K}^3_{1+\epsilon}(\omega_i)} \lesssim N^{-1} |u|_{\mathcal{K}^3_{1+\epsilon}(\omega_i)}.$$

For non-symmetric patches, we use the same procedure in Theorem 3.1. For the most inner layer, i.e., $i \in \mathcal{I}_L$

$$h_i^{-1} s_i \le \|\nabla(u - u_I)\|_{L^2(\omega_i)} h_i^{-1} \|\varphi_i\|_{L^2(\omega_i)} \lesssim N^{-1} \|u\|_{\mathcal{K}^3_{1+\epsilon}(\omega_i)}.$$

For non-symmetric patch in other layers, we use

$$h_i^{-1} s_i \le |\partial_x u - \partial_x u_I|_{L^\infty(\omega_i)} h_i^{-1} \|\varphi_i\|_{L^1(\omega_i)} \lesssim h_i^s |u|_{W^{1+s,\infty}(\omega_i)} |\omega_i|^{1/2},$$

and follow the same procedure in the proof of Theorem 3.1 to obtain

$$\sum_{i=0}^{L-1} \sum_{k \in \mathcal{I}_i} h_k^{-2} s_k^2 \le C N^{-3/2+2\delta} \|u\|^2_{\mathcal{K}^3_{1+\epsilon}(\Omega)}.$$

Put them together, we get the desired estimate. □

We summarize as the following theorem.

Theorem 3.2. *Suppose the grading parameter κ satisfies $\kappa = 2^{-2/\epsilon}$, for some $\epsilon \in (0, \beta)$, and $f \in H^1(\Omega)$. For any $\delta > 0$, there exists a constant $C = C(\delta, \epsilon, \beta, \mathcal{T}_0)$ such that*

$$\|\nabla u - Q_N(\nabla u_N)\|_{L^2(\Omega)} \le C N^{-5/8+\delta} \|f\|_{H^1(\Omega)}.$$

Remark 3.1. The estimate in Theorem 3.1 and thus Theorem 3.2 may be improved by using more refined analysis, e.g., the integral identity over triangles in [7].

3.3. Average-type recovery

In this subsection, we analyze the average type gradient recovery scheme. The proof is similar and thus we only sketch the outline here.

Let us define $R : \nabla \mathbb{V}_N \mapsto \overline{\mathbb{V}}_N \times \overline{\mathbb{V}}_N$

$$R(\nabla u_N)(x_i) = \frac{\sum_{\tau \in \omega_i} |\tau| \nabla u_N|_\tau}{|\omega_i|}, \tag{3.4}$$

where ω_i is the patch of the vertices x_i, i.e., all triangles containing x_i, and $|\cdot|$ is the Lebesgue measure. Then similar superconvergence holds.

Theorem 3.3. *Suppose the grading parameter κ satisfies $\kappa = 2^{-2/\epsilon}$, for some $\epsilon \in (0, \beta)$, and $f \in H^1(\Omega)$. For any $\delta > 0$, there exists a constant $C = C(\delta, \epsilon, \beta, \mathcal{T}_0)$ such that*

$$\|\nabla u - R(\nabla u_N)\|_{L^2(\Omega)} \le C N^{-5/8+\delta} \|f\|_{H^1(\Omega)}.$$

By the triangle inequality

$$\|\nabla u - R(\nabla u_N)\|_{L^2(\Omega)} \le \|\nabla u - R(\nabla u_I)\|_{L^2(\Omega)} + \|R(\nabla u_I - \nabla u_N)\|_{L^2(\Omega)}.$$

As an average operator, it is easy to show R is stable in L^2 norm and thus

$$\|R(\nabla u_I - \nabla u_N)\|_{L^2(\Omega)} \lesssim \|\nabla u_I - \nabla u_N\|_{L^2(\Omega)} \lesssim N^{-5/8+\delta} \|f\|_{H^1(\Omega)}.$$

It remains to estimate the first term. To this end, we need to apply the local analysis in the patch of a triangle τ_i that is

$$U(\tau_i) = \cup_{x_k \in \tau_i} \omega_k.$$

When $u \in \mathcal{P}_1(U(\tau_i))$, ∇u is a constant and thus $R(\nabla u_I) = \nabla u$. By the Bramble-Hilbert lemma, we have the first order estimate

$$\|\nabla u - R(\nabla u_I)\|_{L^2(\tau_i)} \lesssim h_i |u|_{H^2(U(\tau_i))}.$$

When the patch $U(\tau_i)$ is symmetric, the recovery operator will preserve quadratic functions. Namely for $u \in \mathcal{P}_2(U(\tau_i)), R(\nabla u_I) = \nabla u$ in τ_i. Then a second order estimate holds (see e.g. [28])

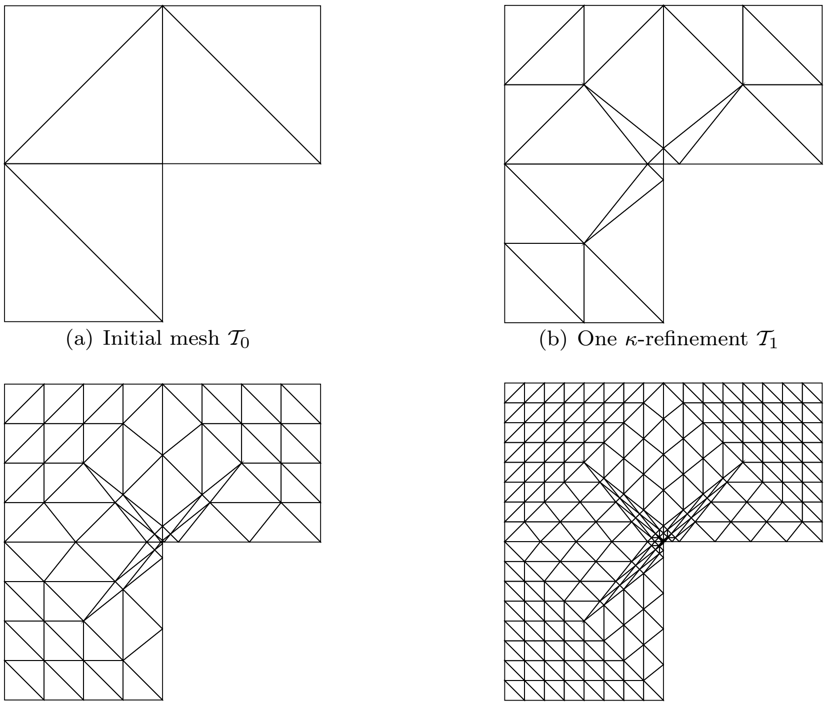
$$\|\nabla u - R(\nabla u_I)\|_{L^2(\tau_i)} \lesssim h_i^2 |u|_{H^3(U(\tau_i))}.$$

For non-symmetric patch, we again use the fact the measure of non-symmetric patches is small. Using the relation of local mesh size and the weight function, we can transform the estimate to weighted Sobolev spaces as before.

4. Numerical Examples

In this section, we shall present numerical examples to support our theoretical results. We shall consider the Poisson equation with the Dirichlet boundary condition:

$$-\Delta u = f, \text{ in } \Omega \quad u = u_D \text{ on } \partial\Omega. \tag{4.1}$$



(a) Initial mesh $\mathcal{T}_0$
(b) One κ-refinement $\mathcal{T}_1$
(c) Two κ-refinements $\mathcal{T}_2$
(d) Three κ-refinements $\mathcal{T}_3$

Fig. 4.1. Graded meshes of the L-shape domain with $\kappa = 0.1$.

The first example is posed on a L-shape domain. Let $\Omega = (-1,1) \times (-1,1) \backslash ([0,1] \times [-1,0])$ be a L-shape domain. We choose u_D and f in (4.1) such that the exact solution u in polar coordinates is

$$u(r,\theta) = r^{\frac{2}{3}} \sin \frac{2}{3}\theta.$$

The second example is a crack problem. Let $\Omega = (-1,1) \times (-1,1) \backslash [0,1]$ be a square domain with a crack. We choose u_D and f in (4.1) such that the exact solution u in polar coordinates is

$$u(r,\theta) = r^{\frac{1}{2}} \sin \frac{\theta}{2} - \frac{1}{4} r^2.$$

We use continuous piecewise linear finite elements to solve these Poisson equations. We adopt LNG_FEM [32] to generate the graded mesh, AFEM@matlab [18] for the solution, and iFEM [16] for the recovered gradient by using the simple average process (3.4). For the L-shape domain, $\beta = 2/3$ and we choose $\kappa = 0.1 < 2^{-2/\beta}$, while we let $\kappa = 0.05 < 2^{-4}$ for the singularity from the crack. We present several graded meshes obtained by κ-refinement in Figure 4.1. The convergence rates of the L-shape problem and the crack problem can be found in Figure 4.2 and in Figure 4.3, respectively.

From Figure 4.2 and 4.3, it is clear that we obtain the optimal convergence rate for $\|\nabla u - \nabla u_N\|_{L^2(\Omega)}$ which is $N^{-1/2}$. The superconvergence rate for $\|\nabla u - R\nabla u_N\|_{L^2(\Omega)}$ is around $N^{-0.65}$ which is very close to the theoretical prediction $-5/8 = -0.625$. We note that the rate of super-closeness $\|\nabla u_I - \nabla u_N\|_{L^2(\Omega)}$ is around $N^{-0.85}$, which is better than our theoretical estimates. See also Remark 3.1.

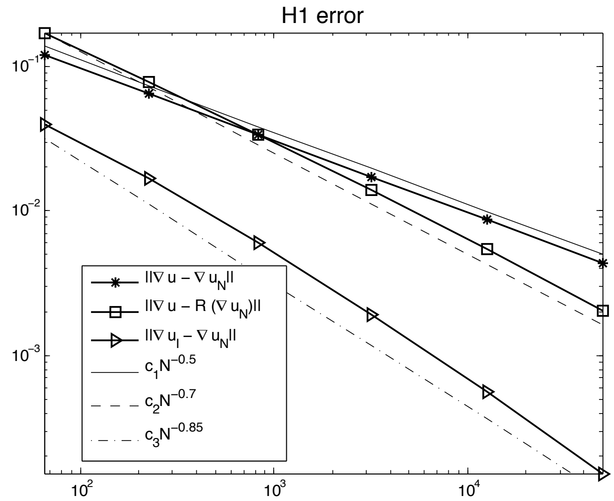


Fig. 4.2. Comparison of convergence rates of the L-shape domain problem.

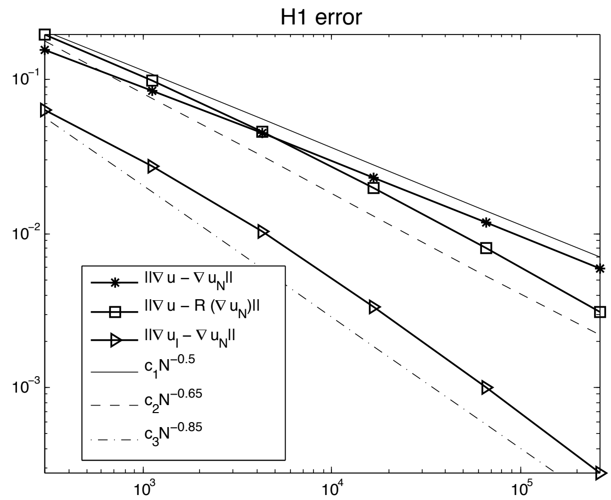


Fig. 4.3. Comparison of convergence rates of the crack problem

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References

[1] R. Adams, Sobolev Spaces, volume 65 of Pure and Applied Mathematics, Academic Press, New York-London, 1975.

[2] M. Ainsworth and J. Oden, A Posteriori Error Estimation in Finite Element Analysis, John Wiley & Sons, Inc., 2000.

[3] I. Babuška, Finite element method for domains with corners, *Computing*, **6** (1970), 264-273.

[4] I. Babuška, R.B. Kellogg, and J. Pitkäranta, Direct and inverse error estimates for finite elements with mesh refinements, *Numer. Math.*, **33**:4 (1979), 447-471.

[5] I. Babuška and T. Strouboulis, The Finite Element Method and its Reliability, Numerical Mathematics and Scientific Computation, Oxford Science Publications, 2001.
[6] C. Băcuţă, V. Nistor, and L.T. Zikatanov, Improving the rate of convergence of 'high order finite elements' on polygons and domains with cusps, *Numer. Math.*, **100**:2 (2005), 165-184.
[7] R. E. Bank and J. Xu, Asymptotically exact a posteriori error estimators, Part I: Grids with superconvergence, *SIAM J. Numer. Anal.*, **41**:6 (2003), 2294-2312.
[8] J. Bergh and J. Lofstrom, Interpolation Spaces, Springer Verlag, New York, 1976.
[9] P. Binev, W. Dahmen, and R. DeVore, Adaptive finite element methods with convergence rates, *Numer. Math.*, **97**:2 (2004), 219-268.
[10] J. Brandts and M. Křížek, Gradient superconvergence on uniform simplicial partitions of polytopes, *IMA J. Numer. Anal.*, **23** (2003), 489-505.
[11] J. Brannick, H. Li, and L.T. Zikatanov, Uniform convergence of the multigrid V-cycle on graded meshes for corner singularities, *Numer. Linear Algebr.*, **15** (2008), 291-306.
[12] J.M. Cascón, C. Kreuzer, R.H. Nochetto, and K.G. Siebert, Quasi-optimal convergence rate for an adaptive finite element method, *Preprint 9, University of Augsburg*, 2007.
[13] C.M. Chen, Superconvergence Theory of Finite Element Methods, Hunan, Science Press, Hunan, China (in Chinese), 2001.
[14] C.M. Chen and Y. Huang, High Accuracy Theory of Finite Element Methods, Hunan, Science Press, Hunan, China (in Chinese), 1995.
[15] L. Chen, Superconvergence of tetrahedral linear finite elements, *International Journal of Numerical Analysis and Modeling*, **3**:3 (2006), 273-282.
[16] L. Chen, *i*FEM: an innovative finite element methods package in MATLAB, Submitted, (2008).
[17] L. Chen and J. Xu, Topics on adaptive finite element methods, In T. Tang and J. Xu, editors, *Adaptive Computations: Theory and Algorithms*, pages 1–31. Science Press, Beijing, 2007.
[18] L. Chen and C.-S. Zhang, AFEM@matlab: a Matlab package of adaptive finite element methods, *Technique Report, Department of Mathematics, University of Maryland at College Park*, 2006.
[19] R. DeVore and G.G. Lorentz, Constructive Approximation, Springer Verlag, New York, 1993.
[20] L. Evans, Partial Differential Equations, Graduate Studies in Mathematics, 1998.
[21] E. Hinton and J.S. Campbell, Local and global smoothing of discontinuous finite element functions using a least squares method, *Int. J. Numer. Meth. Eng.*, **8** (1974), 461-480.
[22] W. Hoffmann, A.H. Schatz, L.B. Wahlbin, and G. Wittum, Asymptotically exact a posteriori estimators for the pointwise gradient error on each element in irregular meshes I: A smooth problem and globally quasi-uniform meshes, *Math. Comput.*, **70** (2001), 897-909.
[23] Y.-Q. Huang, The superconvergence of finite element methods on domains with reentrant corners, In *Finite element methods: superconvergence, post-processing, and a posteriori estimates*, pages 169–182. Marcel Dekker, Inc., New York, NY, USA, 1998.
[24] Y.Q. Huang and Q. Lin, Elliptic boundary value problems in polygonal domains and finite element approximations, *J. Systems Sci. Math. Sci.*, **12**:3 (1992), 263-268.
[25] Y.Q. Huang and Q. Lin, Some estimates of Green functions and their finite element approximations on angular domains, *J. Systems Sci. Math. Sci.*, **14**:1 (1994), 1-8.
[26] V.A. Kondrat′ev, Boundary value problems for elliptic equations in domains with conical or angular points, *Transl. Moscow Math. Soc.*, **16** (1967), 227-313.
[27] M. Křížek and P. Neittaanmäki, On superconvergence techniques, *Acta Appl. Math.*, **9**:3 (1987), 175-198.
[28] M. Křížek and P. Neittaanmäki, Superconvergence phenomenon in the finite element method arising from averaging gradients, *Numer. Math.*, **45** (1984), 105-116.
[29] N. Levine, Superconvergent recovery of the gradient from piecewise linear finite element approximations, *IMA J. Numer. Anal.*, **5** (1985), 407-427.
[30] B. Li and Z.M. Zhang, Analysis of a class of superconvergence patch recovery techniques for linear and bilinear finite elements, *Numer. Meth. Part. D. E.*, **15** (1999), 151-167.

[31] H. Li, A. Mazzucato, and V. Nistor, Analysis of the Finite Element Method for transfission/mixed boundary value problems on general polygonal domians, *CCMA preprint AM319*, 2007.

[32] H.Li and V. Nistor, LNG_FEM: Generating graded meshes and solving elliptic equations on 2-d domains of polygonal structures, *IMA preprint 2172*, 2007.

[33] H. Li and V. Nistor, Analysis of a modified schrödinger operator in 2D: regularity, index, and FEM, *J. Comput. Appl. Math.*, **224** (2009), 320-338.

[34] Z.-C. Li, Global superconvergence of simplified hybrid combinations for elliptic equations with singularities. I. Basic theorem, *Computing*, **65**:1 (2000), 27-44.

[35] Z.C. Li and H.T. Huang, Global superconvergence of simplified hybrid combinations of the ritz-galerkin and fems for elliptic equations with singularities ii. lagrange elements and adini's elements, *Appl. Numer. Math.*, **43**:3 (2002), 253-273.

[36] Q. Lin and J. Xu, Linear finite elements with high accuracy, *J. Comput. Math.*, **3**:2 (1985), 115-133.

[37] G. Raugel, Résolution numérique par une méthode d'éléments finis du problème de Dirichlet pour le laplacien dans un polygone, *C. R. Acad. Sci. Paris Sér. A-B*, **286**:18 (1978), A791-A794.

[38] A. Schatz, I. Sloan, and L. Wahlbin, Superconvergence in finite element methods and meshes that are locally symmetric with respect to a point, *SIAM J. Numer. Anal.*, **33**:2 (1996), 505-521.

[39] R. Stevenson, Optimality of a standard adaptive finite element method, *Found. Comput. Math.*, **7**:2 (2007), 245-269.

[40] H. Wu and Z. Zhang, Can we have superconvergent gradient recovery under adaptive meshes? *SIAM J. Numer. Anal.*, **45**:4 (2007), 1701-1722.

[41] J. Xu and Z.M. Zhang, Analysis of recovery type a posteriori error estimators for mildly structured grids, *Math. Comput.*, (2004), 1139-1152.

[42] N. Yan and A. Zhou, Gradient recovery type a posteriori error estimates for finite element approximations on irregular meshes, *Comput. Method. Appl. M.*, **190**:32-33 (2001), 4289-4299.

[43] Z. Zhang and H.D. Victory, Jr, Mathematical analysis of Zienkiewicz-Zhu's derivative patch recovery technique, *Numer. Meth. Part. D. E.*, **12**:4 (1996), 507-524.

[44] Z. Zhang and J.Z. Zhu, Superconvergence of the derivative patch recovery technique and a posteriori error estimation, In *Modeling, mesh generation, and adaptive numerical methods for partial differential equations (Minneapolis, MN, 1993)*, pages 431–450. Springer, New York, 1995.

[45] Z.M. Zhang and A. Naga, A new finite element gradient recovery method: Superconvergence property, *SIAM J. Sci. Comput.*, **26**:4 (2005), 1192-1213.

[46] Q. Zhu and Q. Lin, Finite element superconvergence theory, Hunan Science Press, Hunan, China, 1989.

[47] O.C. Zienkiewicz and J.Z. Zhu, The superconvergence patch recovery and a posteriori error estimates. Part 1: The recovery techniques, *Int. J. Numer. Meth. Eng.*, **33** (1992), 1331-1364.