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ON STABILITY, ACCURACY, AND FAST SOLVERS FOR FINITE ELEMENT APPROXIMATIONS OF THE AXISYMMETRIC STOKES PROBLEM BY HOOD–TAYLOR ELEMENTS*

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Abstract. We provide a proof of both the stability and the approximation property for the finite element approximations of the axisymmetric Stokes problem by continuous piecewise polynomials of degree $\kappa+1$ for the velocity and continuous piecewise polynomials of degree κ for the pressure with any $\kappa \geq 1$. New techniques are designed so that in this perspective, by a simple transformation, the existing theory developed in three dimensional Cartesian coordinates can be effectively exploited. In fact, this perspective provides a new way of developing theories for the axisymmetric Stokes problems and it can be applied potentially to other problems as well. A simple illustration is provided for the application in the development and analysis of fast solvers for the resulting discrete saddle point problems. Sample numerical experiments have been presented as well to confirm the theoretical results.

Key words. axisymmetric Stokes equation, stability, convergence, preconditioned minimum residual method, multigrid methods

AMS subject classifications. 65N12, 65N30, 65N55

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1. Introduction.

Many classical problems in fluid dynamics can be modeled by Stokes equations such as a sphere falling through a viscous Newtonian fluid as excellently described in [12]. In recent years, it is of tremendous interest to study the complex fluids flows and complex fluids are typically classified as creeping flows [4]. In particular, when the domain of interest and the data are axisymmetric, the complex fluids are typically axisymmetric. Furthermore, many simulations in these classes of flow equations can be performed by the solution of the Stokes equation together with the solution to some transport equation that governs the stress fields [15]. Therefore, it is extremely useful to consider the axisymmetric form of the Stokes equation in practice since it can afford significant reductions in computation time without loss of the solution's reliability in general [16]. Therefore, the importance of handling the Stokes equation in a stable and efficient way cannot be emphasized too much. In fact, the use of the axisymmetric form of the Stokes equation is a common practice and the finite element discretization of the Stokes equation is popular in engineering. The most commonly used finite element pairs are, perhaps, the Hood–Taylor finite elements, which consists of continuous piecewise polynomials of degree $\kappa+1$ for the velocity and continuous piecewise polynomials of degree κ for the pressure, especially with $\kappa = 1$. On the other hand, it seems that the important issue in the stability analysis, the so-called inf-sup condition, has not been clearly handled for axisymmetric

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Stokes equations in literature. We note that some discussion can be found in the series of papers by Tabata [23, 24], especially for $P2/P1$ finite elements. We note that it is very recent that the pair of finite elements, the so-called $P1\text{iso}P2/P1$ elements, is proven to be stable for axisymmetric Stokes equations [2]. We also refer readers to the work by Ying [26] for the proof of stability of $P2/P0$ elements.

In this paper, we shall establish that the general Hood–Taylor finite elements of any degree $\kappa \geq 1$ are stable for axisymmetric Stokes equations. Our motivation and idea came from the work by Boffi [5], who proved similar results for the three dimensional Stokes equation using the arguments from Stenberg [22]. The framework we developed here in this paper is therefore based on the full three dimensional considerations. This is the main novelty in this work. The advantage in this approach is that many known theories in Cartesian coordinate systems can be effectively exploited. In fact, a simple illustration is provided in this paper in terms of the development and analysis of the fast solver for the resulting saddle point problems. In particular, to the best of the authors' knowledge, the stability of the higher order Hood–Taylor finite elements for axisymmetric Stokes equation is missing in literature. A few numerical experiments have been provided, especially for the lowest Hood–Taylor finite elements to confirm theoretical results, which include the convergence of the methods and the optimality of the solvers based on the preconditioned minimum residual iterative methods.

The remainder of this paper is organized as follows. In section 1.1, we present notation as preliminary to the presentation. In section 2, we present the model equations and the well-posedness. The stability and approximation properties of the Hood–Taylor finite elements are established as well. In section 3, we illustrate how our framework can be effectively used to design and analyze the fast solver for the resulting saddle point problems. Sample numerical experiments have been presented in section 4. This paper concludes with several remarks in section 5. Many technical results and proofs are included in the appendix.

1.1. Preliminaries. Following Bernardi, Dauge, and Maday [3], for a generic point $\mathbf{x}$ in $\mathbb{R}^3$, we use both Cartesian coordinates $\mathbf{x} = (x, y, z)$ and cylindrical coordinates denoted by (r, θ, z). In $\mathbb{R}^2$, we shall use the restricted coordinates (r, z) and we define the half-space $\mathbb{R}^2_+$ as the set of points in $\mathbb{R}^2$ with positive r. Let Ω denote a bounded meridian domain contained in $\mathbb{R}^2_+$ and denote by $\breve{\Omega} \subset \mathbb{R}^3$ the axisymmetric domain obtained by rotating Ω around the axis $r = 0$. We denote by Γ_0 the interior of the part of the boundary $\partial\Omega$ contained in the axis $r = 0$ and set $\Gamma = \partial\Omega / \Gamma_0$. Let $\mathcal{R}_\eta$ denote the rotation with angle η with respect to the axis $r = 0$ [3]. The unit outward normal vector $\breve{\mathbf{n}}$ on $\partial\breve{\Omega}$ is obtained by rotating the unit outward normal vector $\mathbf{n}$ on Γ. We use the standard Sobolev space notation for the domain $\breve{\Omega}$. The symbol $L^2(\breve{\Omega})$ denotes the space of square integrable functions for the measure of $dxdydz = d\mathbf{x}$ and $H^s(\breve{\Omega})$ for any real s, which denotes the standard Sobolev space [1]. The space $H^s_0(\breve{\Omega})$ will denote the subspace of $H^s(\breve{\Omega})$ with zero boundary values. We shall use boldfaced letters to denote spaces for vector fields by $\mathbf{L}^2(\breve{\Omega})$ and $\mathbf{H}^s(\breve{\Omega})$. We say that a function $\breve{w}$ is invariant by the rotation or axisymmetric if it satisfies that $\breve{w} \circ \mathcal{R}_\eta = \breve{w}$ for any $\eta \in [-\pi, \pi]$. On the other hand, a vector field $\breve{\mathbf{w}}$ is said to be invariant by the rotation or axisymmetric if it satisfies that $\breve{\mathbf{w}} = \mathcal{R}_{-\eta}\breve{\mathbf{w}} \circ \mathcal{R}_\eta$ for any $\eta \in [-\pi, \pi]$. We shall denote the subspace of $H^s(\breve{\Omega})$ (or $\mathbf{H^s}(\breve{\boldsymbol{\Omega}})$) which consists of functions that are axisymmetric by $\breve{H}^s(\breve{\Omega})$ (or $\breve{\mathbf{H}}^s(\breve{\Omega})$). By the change of variables, the measure $dxdydz$ can be transformed into $rdrd\theta dz$; therefore, it is natural to study weighted Sobolev spaces on Ω associated with the measure $rdrdz$. The space $L^2_\alpha(\Omega)$ is defined as the set of measurable functions w such that

$$\|w\|_{0,\alpha,\Omega} = \left(\int_\Omega w^2(r,z) r^\alpha drdz\right)^{1/2} < +\infty. \tag{1.1}$$

For any positive integer m, $H^m_\alpha(\Omega)$ is the space of functions w in $L^2_\alpha(\Omega)$ such that their partial derivatives of order $\le m$ belong to $L^2_\alpha(\Omega)$. The space $H^m_\alpha(\Omega)$ is provided with the seminorm $|\cdot|_{H^m_\alpha(\Omega)}$ and the norm $\|\cdot\|_{H^m_\alpha(\Omega)}$ given as follows:

$$|w|_{m,\alpha,\Omega} = \left(\sum_{\ell=0}^m \|\partial_r^\ell \partial_z^{m-\ell} w\|^2_{L^2_\alpha(\Omega)}\right)^{1/2} \quad \text{and} \quad \|w\|_{m,\alpha,\Omega} = \left(\sum_{\ell=0}^m |w|_{H^m_\alpha(\Omega)}\right)^{1/2}.$$

We shall denote by $H^m_{\alpha,0}$ the space of functions that belong to $H^m_\alpha(\Omega)$ and take zero values at Γ. We define additional weighted space $V^1_1(\Omega)$ by $V^1_1(\Omega) = \{w \in H^1_1(\Omega) : w \in L^2_{-1}(\Omega)\}$ with the norm defined by $\|w\|_{V^1_1(\Omega)} = \left(|w|^2_{1,1,\Omega} + \|w\|^2_{0,-1,\Omega}\right)^{1/2}$. We shall also introduce the space $V^1_{1,0} = V^1_1 \cap H^1_{1,0}(\Omega)$. It is well known that all functions in $V^1_1(\Omega)$ have a null trace on Γ_0 [19]. We shall denote $H^s_0(\breve{\Omega}), \breve{H}^s_0(\breve{\Omega}), \mathbf{H}^s_0(\breve{\Omega})$, and $\breve{\mathbf{H}}^s_0(\breve{\Omega})$ by the space of functions that have zero trace on $\partial\breve{\Omega}$ or Γ. We note that the aforementioned normed spaces are Hilbert spaces with corresponding inner products. In particular, we shall often use the inner product for the space $L^2_\alpha(\Omega)$, denoted by $(\cdot,\cdot)_{0,\alpha,\Omega}$ and given as $(u,v)_{0,\alpha,\Omega} = \int_\Omega uvr^\alpha drdz$. The norms $\|\cdot\|_{m,\alpha,\Omega}$ or $|\cdot|_{m,\alpha,\Omega}$ shall be used simply as $\|\cdot\|_{m,\alpha}$ or $|\cdot|_{m,\alpha}$. However, for cases when these norms are used in the subset, say $\Sigma \subset \Omega$, we shall use $\|\cdot\|_{m,\alpha,\Sigma}$ or $|\cdot|_{m,\alpha,\Sigma}$. The same convention will also be used for the inner products and for the full three dimensional cases. Throughout this paper, for a domain, say $K \subset \Omega$, we denote

$$r_{\min}(K) = \min\{r : \forall (r,z) \in K\} \quad \text{and} \quad r_{\max}(K) = \max\{r : \forall (r,z) \in K\}.$$

Any axisymmetric function $\breve{v}$ can be completely characterized by the function v defined by $v(r,z) = \breve{v}(x,y,z)$. We also note that any vector field $\breve{\mathbf{v}} = (v_1, v_2, v_3)$ can be associated with its radial component v_r, the angular component v_θ, and the axial component v_z in the cylindrical coordinate system; i.e., $v_r = v_1\cos\theta + v_2\sin\theta$, $v_\theta = -v_1\sin\theta + v_2\cos\theta$, $v_z = v_3$, and it holds that the fact that $\breve{\mathbf{v}}$ is axisymmetric is equivalent to the fact that v_r, v_θ, and v_z are axisymmetric. Therefore, any axisymmetric vector fields can be completely characterized by the functions defined only on Ω. We note that we can associate any given function $\overline{q}$ and vector fields $\overline{\mathbf{v}} = (v_r, v_\theta, v_z)^t$ defined on Ω, with $\breve{q}$ and $\breve{\mathbf{v}}$ defined on $\breve{\Omega}$ as follows: $\breve{q}(x,y,z) = \overline{q}(r,z)$ and $\breve{\mathbf{v}} = (v_r\cos\theta - v_\theta\sin\theta, v_r\sin\theta + v_\theta\cos\theta, v_z)^t$.

2. Stokes equations with axisymmetric data.

In this section, we introduce the Stokes equations formulated in the axisymmetric domain $\breve{\Omega}$ with axisymmetric data $\breve{\mathbf{f}}$ and establish the well-posedness in a different framework from the available literature such as Bernardi, Dauge, and Maday [3].

2.1. Model description.

A creeping flow through a domain $\breve{\Omega}$ can be mathematically modeled by the axisymmetric Stokes equation under the assumption that the body force $\overline{\mathbf{f}} = (f_r, f_\theta, f_z)^t$ is axisymmetric. Namely,

$$\begin{aligned}
-\partial_r^2 u_r - r^{-1}\partial_r u_r - \partial_z^2 u_r + r^{-2}u_r + \partial_r p &= f_r,\\
-\partial_r^2 u_\theta - r^{-1}\partial_r u_\theta - \partial_z^2 u_\theta + r^{-2}u_\theta &= f_\theta,\\
-\partial_r^2 u_z - r^{-1}\partial_r u_z - \partial_z^2 u_z + \partial_z p &= f_z,\\
-\partial_r u_r - r^{-1}u_r - \partial_z u_z &= 0,
\end{aligned}$$

which is equivalent to the following: for $\breve{\mathbf{f}} \in \breve{\mathbf{L}}^2(\breve{\Omega})$, find $\breve{\mathbf{u}}$ and $\breve{p}$ such that

$$-\Delta\breve{\mathbf{u}} + \nabla\breve{p} = \breve{\mathbf{f}} \text{ in } \breve{\Omega}, \quad \nabla\cdot\breve{\mathbf{u}} = 0 \text{ in } \breve{\Omega}, \quad \breve{\mathbf{u}} = 0 \text{ on } \partial\breve{\Omega}. \tag{2.1}$$

By using the Sobolev spaces introduced in section 1.1, we now provide the variational formulation of the aforementioned (2.1) as follows: find $(\breve{\mathbf{u}}, \breve{p}) \in \breve{\mathbf{H}}_0^1(\breve{\Omega}) \times \breve{L}_0^2(\breve{\Omega})$ such that

$$a(\breve{\mathbf{u}}, \breve{\mathbf{v}}) + b(\breve{\mathbf{v}}, \breve{p}) = \langle \breve{\mathbf{f}}\breve{\mathbf{v}} \rangle \ \forall \breve{\mathbf{v}} \in \breve{\mathbf{H}}_0^1(\breve{\Omega}) \quad \text{and} \quad b(\breve{\mathbf{u}}, \breve{q}) = 0 \ \forall \breve{q} \in \breve{L}^2(\breve{\Omega}), \tag{2.2}$$

where $a(\breve{\mathbf{u}}, \breve{\mathbf{v}}) = \int_{\breve{\Omega}} \nabla\breve{\mathbf{u}} : \nabla\breve{\mathbf{v}}\, d\mathbf{x}$ and $b(\breve{\mathbf{u}}, \breve{q}) = -\int_{\breve{\Omega}} \breve{q}\nabla\cdot\breve{\mathbf{u}} d\mathbf{x}$. It is well known that (2.2) is well posed [9]. It is established that the aforementioned (2.2) admits a unique axisymmetric solution in [3], where a dimensional reduction technique is exploited. We shall establish the result in a different manner. Namely, instead of reducing (2.2) into a two dimensional cylindrical coordinate setting, we shall demonstrate that for any given axisymmetric function $\breve{p}$, there exists an axisymmetric vector field $\breve{\mathbf{u}}$ that satisfies $\nabla\cdot\breve{\mathbf{u}} = \breve{p}$. This shall establish the inf-sup condition for the equation of our interest and proves the same results.

THEOREM 2.1. *Given $\breve{q} \in \breve{L}^2(\breve{\Omega})$, there exists $\breve{\mathbf{u}} \in \breve{\mathbf{H}}_0^1(\breve{\Omega})$ such that $\nabla\cdot\breve{\mathbf{u}} = \breve{q}$.*

In order to prove the existence of such a vector field, we shall recall the following well-known result [11].

LEMMA 2.2. *For any $q \in L^2(\breve{\Omega})$ with $\int_{\breve{\Omega}} q\, dx = 0$, there exists a function $\mathbf{u} \in \mathbf{H}_0^1(\breve{\Omega})$ such that $\nabla\cdot\mathbf{u} = q$ with $\|\mathbf{u}\|_1 \lesssim \|q\|_0$.*

Proof of Theorem 2.1. According to Lemma 2.2, for a given smooth function $\breve{q}$, we can find a vector field $\mathbf{u}$, which is not necessarily axisymmetric, such that $\mathbf{u}|_{\partial\Omega} = 0$ and $\nabla\cdot\mathbf{u} = \breve{q}$. Now, for any $\eta \in [-\pi, \pi)$, we define $\mathbf{u}_\eta$ and $\breve{\mathbf{u}}$, respectively, by

$$\mathbf{u}_\eta = \mathcal{R}_{-\eta}\mathbf{u} \circ \mathcal{R}_\eta \quad \text{and} \quad \breve{\mathbf{u}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbf{u}_\eta \, d\eta.$$

By definition, $\breve{\mathbf{u}}$ is axisymmetric and the following relation holds true:

$$\nabla\cdot\breve{\mathbf{u}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \nabla\cdot\mathbf{u}_\eta \, d\eta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \breve{q}\circ\mathcal{R}_\eta(\mathbf{x})\, d\eta = \breve{q}(\mathbf{x}).$$

Furthermore, it is easy to see that the following inequalities hold true: $\|\breve{\mathbf{u}}\|_1^2 \lesssim \|\mathbf{u}\|_1^2 \lesssim \|\breve{q}\|_0$. The standard density argument completes the proof. ◻

As an auxiliary problem, we also consider the following vector Poisson's equation: Find $\breve{\mathbf{u}} \in \breve{\mathbf{H}}_0^1(\breve{\Omega})$ such that

$$a(\breve{\mathbf{u}}, \breve{\mathbf{v}}) = \langle \breve{\mathbf{f}}, \breve{\mathbf{v}} \rangle \quad \forall \breve{\mathbf{v}} \in \breve{\mathbf{H}}_0^1(\breve{\Omega}). \tag{2.3}$$

Similarly to Theorem 2.1, we can also establish the following fact.

LEMMA 2.3. *For $\breve{\mathbf{f}} \in \breve{\mathbf{L}}^2(\breve{\Omega})$, (2.3) admits a unique solution $\breve{\mathbf{u}} \in \breve{\mathbf{H}}_0^1(\breve{\Omega})$.*

2.2. "Toroid" finite elements and interpolation operators. In this section, we shall introduce finite elements that are defined on $\breve{\Omega}$ which can be reduced to the standard finite elements when the dimensional reduction is applied for the computational purpose. This will be the main tool in providing the new angle that can be used to exploit the existing theories available in the Cartesian coordinate system in a very natural way when developing relevant theories for the axisymmetric cases. We begin by triangulating Ω using the standard shape regular finite elements [6] denoted by $\mathcal{T}_h = \{\tau_k\}_k$. We then define for each k, $\breve{\tau}_k$ by

$$\breve{\tau}_k = \{(r\cos\theta, r\sin\theta, z) : (r,z) \in \tau_k \text{ and } \theta \in [0, 2\pi]\}. \tag{2.4}$$

We shall then collect $\breve{\tau}_k$ to form the triangulation $\breve{\mathcal{T}}_h = \{\breve{\tau}_k\}_k$ of $\breve{\Omega}$. Upon the construction of the triangulation, we shall now build the finite element functions again from the standard piecewise finite element, say $p(r,z)$ defined on $\mathcal{T}_h$. Note that we shall restrict our concern to only the continuous functions and denote the space of polynomials of degree κ on $\Sigma \in \Omega$ by $\mathcal{P}^\kappa(\Sigma)$. The symbol $\breve{\mathcal{P}}^\kappa(\breve{\Sigma})$ denotes the three dimensional representation of the space $\mathcal{P}^\kappa(\Sigma)$. We first define the standard (continuous) piecewise polynomials of degree $\kappa \geq 0$ on $\mathcal{T}_h$ and define the corresponding finite elements to the triangulation $\breve{\mathcal{T}}_h$ of degree $\kappa \geq 0$ by $\breve{p}(x,y,z) = p(r,z)$. To be more precise, we denote the piecewise polynomial functions of degree $\kappa \geq 0$ defined on τ by

$$P_h^\kappa = \left\{ q_h \in \mathcal{C}^0(\Omega) : q_h|_\tau \in \mathcal{P}^\kappa(\tau) \quad \forall \tau \in \mathcal{T}_h \right\}. \tag{2.5}$$

The finite element space defined on the "toroid" elements $\breve{\tau} \in \breve{\mathcal{T}}_h$ can be defined as follows:

$$\breve{P}_h^\kappa = \left\{ \breve{q}_h \in \breve{\mathcal{C}}^0(\breve{\Omega}) : \breve{q}_h|_{\breve{\tau}} \in \breve{\mathcal{P}}^\kappa(\breve{\tau}) \quad \forall \breve{\tau} \in \breve{\mathcal{T}}_h \right\}. \tag{2.6}$$

We shall use the notation that $h = \max_k \operatorname{diam}(\tau_k)$ and denote nodes of the triangulation $\mathcal{T}_h$ by $\{x_i\}_i$. From the aforementioned "toroid" finite elements, we shall define axisymmetric Hood–Taylor finite elements as follows: for fixed $\kappa \geq 1$, we define the finite element function space for the pressure by

$$\breve{S}_h = \left\{ \breve{q} \in \breve{P}_h^\kappa : \int_{\breve{\Omega}} \breve{q} d\breve{\mathbf{x}} = 0 \right\}, \tag{2.7}$$

and the finite element space for the velocity fields by

$$\begin{aligned} \breve{\mathbf{V}}_h = \{ & \breve{\mathbf{v}}_h = (\breve{v}_r\cos\theta - \breve{v}_\theta\sin\theta, \breve{v}_r\sin\theta + \breve{v}_\theta\cos\theta, \breve{v}_z)^t \\ & \text{with } v_r, v_\theta|_{\partial\Omega} = 0, v_z|_\Gamma = 0 \text{ with } v_r, v_\theta, v_z \in P_h^{\kappa+1}\}. \end{aligned} \tag{2.8}$$

The discrete weak formulation of (2.2) reads as follows: find $(\breve{\mathbf{u}}_h, \breve{p}_h) \in \breve{\mathbf{V}}_h \times \breve{S}_h$ such that

$$a(\breve{\mathbf{u}}_h, \breve{\mathbf{v}}_h) + b(\breve{\mathbf{v}}_h, \breve{p}_h) = \langle \breve{\mathbf{f}}, \breve{\mathbf{v}}_h \rangle \quad \forall \breve{\mathbf{v}}_h \in \breve{\mathbf{V}}_h, \quad b(\breve{\mathbf{u}}_h, \breve{q}_h) = 0 \quad \forall \breve{q}_h \in \breve{S}_h. \tag{2.9}$$

It is easy to notice that (2.9) is reduced to the standard axisymmetric formulation of the Stokes equation [2] by coordinate changes into the cylindrical coordinate in terms of the finite element pairs $\overline{\mathbf{V}}_h$ and $\overline{S}_h$ defined by

$$\overline{\mathbf{V}}_h = \left\{ (v_r, v_\theta, v_z)^t : v_r, v_\theta|_{\partial\Omega}, v_z|_\Gamma = 0, v_r, v_\theta, v_z \in P_h^{\kappa+1} \right\} \text{ and } \overline{S}_h = P_h^\kappa \cap L^2_{0,r}(\Omega).$$

Namely, the axisymmetric formulation of the Stokes equation reads as follows: find $\overline{\mathbf{u}}_h = (u_r, u_\theta, u_z)^t \in \overline{\mathbf{V}}_h$ and $\overline{p}_h \in \overline{S}_h$ such that for all $\overline{\mathbf{v}}_h \in \overline{\mathbf{V}}_h$ and $\overline{q}_h \in \overline{S}_h$,

$$\langle \mathcal{A}\overline{\mathbf{u}}_h, \overline{\mathbf{v}}_h \rangle + \langle \mathcal{B}\overline{p}_h, \overline{\mathbf{v}}_h \rangle = \langle \overline{\mathbf{f}}, \overline{\mathbf{v}}_h \rangle \quad \text{and} \quad \langle \mathcal{B}^*\overline{\mathbf{u}}_h, \overline{q}_h \rangle = 0, \tag{2.10}$$

where $\langle \mathcal{A}\overline{\mathbf{u}}, \overline{\mathbf{v}}\rangle = (u_r, v_r)_{V_1^1} + (u_\theta, v_\theta)_{V_1^1} + (u_z, v_z)_{H_1^1}$ and $\langle \mathcal{B}p, \mathbf{v}\rangle = (\partial_r p, v_r)_{0,1} + (\partial_z p, v_z)_{0,1}$. Equation (2.10) is used for practical computations, thus the main question of the stability has been posed to show that

$$\sup_{0\neq \mathbf{v}\in\overline{\mathbf{V}}_h} \frac{(\partial_r p, v_r)_{0,1} + (\partial_z p, v_z)_{0,1}}{\|v_r\|_{V_1^1} + \|v_\theta\|_{V_1^1} + |v_z|_{H_1^1}} \gtrsim \|p\|_{0,1} \quad \forall p \in \overline{S}_h. \tag{2.11}$$

Our strategy is to show rather the equivalent condition to (2.11) given as follows:

$$\sup_{0\neq \check{\mathbf{v}}\in\check{\mathbf{V}}_h} \frac{(\nabla\cdot\check{\mathbf{v}}, \check{p})_0}{|\check{\mathbf{v}}|_1} \gtrsim \|\check{p}\|_0 \quad \forall \check{p} \in \check{S}_h. \tag{2.12}$$

2.2.1. Interpolation operator for velocity fields. The aim of this section is to construct the interpolation operator $\check{\mathbf{\Pi}}_h : \check{\mathbf{H}}_0^1(\check{\Omega}) \mapsto \check{\mathbf{V}}_h$ such that for $\kappa \geq 1$, $\|\check{\mathbf{u}} - \check{\mathbf{\Pi}}_h\check{\mathbf{u}}\|_1 \lesssim h^{\kappa+1}\|\check{\mathbf{u}}\|_{\kappa+2}$ and $\|\check{\mathbf{\Pi}}_h\check{\mathbf{u}}\|_1 \lesssim \|\check{\mathbf{u}}\|_1$. We note that any axisymmetric vector field $\check{\mathbf{v}} \in \check{\mathbf{H}}_0^1(\check{\Omega})$ can be represented by $\overline{\mathbf{v}} = (v_r, v_\theta, v_z)^t \in V_{1,0}^1 \times V_{1,0}^1 \times H_{1,0}^1$ as $\check{\mathbf{v}} = (v_r\cos\theta - v_\theta\sin\theta, v_r\sin\theta + v_\theta\cos\theta, v_z)^t$. For the construction of $\check{\mathbf{\Pi}}_h$, we shall construct two interpolation operators, say $\Pi_h^+ : H_{1,0}^1 \mapsto H_{1,0}^1 \cap P_h^{\kappa+1}$ and $\Pi_h^- : V_{1,0}^1 \mapsto V_{1,0}^1 \cap P_h^{\kappa+1}$, so that they satisfy both the stability and the approximation property; see Theorems 2.4 and 2.5 below. We then define $\check{\mathbf{\Pi}}_h\check{\mathbf{u}}$ by

$$\check{\mathbf{\Pi}}_h\check{\mathbf{u}} = \begin{pmatrix} (\Pi_h^- u_r)\cos\theta - (\Pi_h^- u_\theta)\sin\theta \\ (\Pi_h^- u_r)\sin\theta + (\Pi_h^- u_\theta)\cos\theta \\ \Pi_h^+ u_z \end{pmatrix}. \tag{2.13}$$

The operator $\check{\mathbf{\Pi}}_h$ will then be shown to satisfy the desired properties.

We begin to construct $\Pi_h^+ : H_{1,0}^1(\Omega) \mapsto H_{1,0}^1 \cap P_h^{\kappa+1}(\Omega)$. The technical issue is to preserve the zero boundary condition on Γ. Following the idea of Scott and Zhang [21], for the node x_i on the boundary $\bar{\Gamma}$, we can choose an edge on $\bar{\Gamma}$ associated with it, denoted by $e(x_i)$, and choose a triangle τ_i that contains $e(x_i)$. We require that if $x_i \in \overline{e}(x_i)$ does not belong to the z-axis, neither $e(x_i)$ nor τ_i intersects the z-axis. We define $\pi_i : H_1^1(\tau_i) \mapsto \mathcal{P}^{\kappa+1}(e(x_i))$ as follows:

$$\int_{e(x_i)} \pi_i v\psi r dr dz = \int_{e(x_i)} v\psi r dr dz \quad \forall v \in H_1^1(\tau_i), \forall\psi \in \mathcal{P}^{\kappa+1}(e(x_i)). \tag{2.14}$$

For a node x_i, which is not on the boundary $\bar{\Gamma}$, we associate it with a triangle $\tau(x_i)$ such that $x_i \in \overline{\tau}(x_i)$. We define $\pi_i : H_1^1(\tau(x_i)) \mapsto \mathcal{P}^{\kappa+1}(\tau(x_i))$ as follows:

$$\int_{\tau(x_i)} \pi_i v\psi r dr dz = \int_{\tau(x_i)} v\psi r dr dz \quad \forall v \in H_1^1(\tau(x_i)), \forall\psi \in \mathcal{P}^{\kappa+1}(\tau(x_i)). \tag{2.15}$$

The interpolation operator $\Pi_h^+ : H_{1,0}^1(\Omega) \mapsto P_h^{\kappa+1}(\Omega)$ can be defined as follows:

$$\Pi_h^+ v := \sum_i \pi_i v(x_i)\phi_i \quad \forall v \in H_{1,0}^1(\Omega), \tag{2.16}$$

where $\{\phi_i\}_i$ is the nodal basis for the space $P_h^{\kappa+1}(\Omega)$. We shall now state the main property of the operator Π_h^+. The proof is given in the appendix.

THEOREM 2.4. *For $\kappa \geq 1$, the operator Π_h^+ is the projection on $P_h^{\kappa+1}$, i.e., $\Pi_h^+ v = v \forall v \in P_h^{\kappa+1}$ and $\|\Pi_h^+ v\|_{1,1} \lesssim \|v\|_{1,1}$ $\forall v \in H_{1,0}^1(\Omega)$. Furthermore, for $\ell = 0, 1$, it holds true that*

$$|v - \Pi_h^+ v|_{\ell,1} \lesssim h^{\kappa+2-\ell} \|v\|_{\kappa+2,1} \quad \forall v \in H_r^{\kappa+2}(\Omega). \tag{2.17}$$

We need an additional interpolation operator $\Pi_h^- : V_{1,0}^1(\Omega) \mapsto P_h^{\kappa+1}(\Omega)$ to take into account the fact that any functions in $V_{1,0}^1(\Omega)$ have the zero null on Γ_0. This can be done by a simple modification of the definition of Π_h^+ and it should be done in such a way that for $v \in V_{1,0}^1(\Omega)$, $\Pi_h^- v$ takes zero values on Γ_0. For all nodes $\{x_i\}_i$ away from the z-axis, we define $\Pi_h^- v$ by $(\Pi_h^- v)(x_i) = (\Pi_h^+ v)(x_i)$. For nodes x_i on the z-axis, we shall choose an edge $e(x_i)$ containing x_i so that it belongs entirely on the z-axis. We let $\tau_i \in \mathcal{T}_h$ be a triangle that contains $e(x_i)$. We define $\pi_{i,r} : V_1^1(\tau_i) \mapsto \mathcal{P}^{\kappa+1}(e(x_i))$ by

$$\int_{e(x_i)} \pi_{i,r} v \psi \, dr dz = \int_{e(x_i)} v \psi \, dr dz \quad \forall \psi \in \mathcal{P}^{\kappa+1}(e(x_i)). \tag{2.18}$$

Similarly to Π_h^+, the interpolation operator $\Pi_h^- : V_{1,0}^1(\Omega) \mapsto P_h^{\kappa+1}(\Omega)$ can be defined as follows:

$$\Pi_h^- v = \sum_{\{i, x_i \notin \{r=0\}\}} \Pi_h^+ v(x_i) \phi_i + \sum_{\{i, x_i \in \{r=0\}\}} \pi_{i,r} v(x_i) \phi_i \quad \forall v \in V_{1,0}^1(\Omega). \tag{2.19}$$

In fact, $\pi_{i,r} v(x_i) = 0$, and therefore, the interpolation operator Π_h^- preserves the boundary condition everywhere on $\partial\Omega$. We shall now state the analogous results for Π_h^-. The proof is given in the appendix.

THEOREM 2.5. *For $\kappa \geq 1$, the operator $\Pi_h^- : V_{1,0}^1(\Omega) \mapsto P_h^{\kappa+1}(\Omega)$ satisfies $\|\Pi_h^- v\|_{V_1^1} \lesssim \|v\|_{V_1^1} \forall v \in V_{1,0}^1$, and*

$$\|v - \Pi_h^- v\|_{V_1^1} \lesssim h^{\kappa+1}(\|v\|_{\kappa+2,1} + \|v\|_{0,-1}) \quad \forall v \in H_1^{\kappa+2}(\Omega) \cap L_{-1}^2(\Omega). \tag{2.20}$$

We are in position to prove the main properties of the operator $\breve{\mathbf{\Pi}}_h$. We begin our discussion by noting that for $\breve{\mathbf{u}} \in \breve{\mathbf{H}}^1(\breve{\Omega})$,

$$\|\breve{\mathbf{u}}\|_0^2 = 2\pi \left(\|u_r\|_{0,1}^2 + \|u_\theta\|_{0,1}^2 + \|u_z\|_{0,1}^2 \right), \tag{2.21}$$

$$|\breve{\mathbf{u}}|_1^2 = 2\pi \left(|u_r|_{V_1^1}^2 + |u_\theta|_{V_1^1}^2 + |u_z|_{1,1}^2 \right), \tag{2.22}$$

where $\breve{\mathbf{u}} = (u_r \cos\theta - u_\theta \sin\theta, u_r \sin\theta + u_\theta \cos\theta, u_z)^t$. The following theorem can then be easily derived from the aforementioned Theorems 2.4 and 2.5.

THEOREM 2.6. *The operator $\breve{\mathbf{\Pi}}_h$ satisfies the following properties: that for $\kappa \geq 1$,*

$$\|\breve{\mathbf{u}} - \breve{\mathbf{\Pi}}_h \breve{\mathbf{u}}\|_1 \lesssim h^{\kappa+1} \|\breve{\mathbf{u}}\|_{\kappa+2} \quad \text{and} \quad \|\breve{\mathbf{\Pi}}_h \breve{\mathbf{u}}\|_1 \lesssim \|\breve{\mathbf{u}}\|_1. \tag{2.23}$$

Proof. The approximation property follows from the identities (2.21) and (2.22) and Theorems 2.4 and 2.5,

$$\begin{aligned}
&\|\breve{\mathbf{u}} - \breve{\mathbf{\Pi}}_h \breve{\mathbf{u}}\|_1 \lesssim \|u_r - \Pi_h^- u_r\|_{V_1^1} + \|u_\theta - \Pi_h^- u_\theta\|_{V_1^1} + \|u_z - \Pi_h^+ u_z\|_{1,1} \\
&\lesssim h^{\kappa+1} \left(\|u_r\|_{\kappa+2,1} + \|u_r\|_{0,-1} + \|u_\theta\|_{\kappa+2,1} + \|u_\theta\|_{0,-1} + \|u_z\|_{\kappa+2,1} \right) \lesssim h^{\kappa+1} \|\breve{\mathbf{u}}\|_{\kappa+2}.
\end{aligned}$$

The stability result can be shown similarly. This completes the proof. ∎

2.2.2. Approximation property of pressure fields. The aim of this section is to establish the approximation property of the pressure finite element spaces.

THEOREM 2.7. *The following optimal approximation property holds true:*

$$\inf_{\breve{\phi}_h \in \breve{S}_h} \|\breve{q} - \breve{\phi}_h\|_0 \lesssim h^{\kappa+1}\|\breve{q}\|_{\kappa+1}, \quad \forall \breve{q} \in \breve{H}^{\kappa+1}(\breve{\Omega}). \tag{2.24}$$

Proof. We provide the estimate on $\inf_{\breve{\phi}_h \in \breve{S}_h} \|\breve{q} - \breve{\phi}_h\|_0$. We note that it is enough to establish the estimate for $\inf_{\phi_h \in S_h} \|q - \phi_h\|_{0,1}$. On the other hand, we note that $\inf_{\phi_h \in S_h} \|q - \phi_h\|_{0,1} = \|q - q_h\|_{0,1}$, where q_h is the $(\cdot,\cdot)_{0,1}$ projection of q onto P_h^κ. We observe that q_h satisfies that

$$(q_h, p_h)_{0,1} = (q, p_h)_{0,1} \quad \forall p_h \in P_h^\kappa. \tag{2.25}$$

Therefore, by choosing $p_h = 1$, we note that q_h belongs to S_h, since $(q_h, 1)_{0,1} = (q, 1)_{0,1} = 0$. Now, in particular, we have that

$$\|q - q_h\|_{0,1} \le \|q - \Pi_h^+ q\|_{0,1} \lesssim h^{\kappa+1}\|q\|_{\kappa+1,1} \lesssim h^{\kappa+1}\|\breve{q}\|_{\kappa+1}.$$

This completes the proof. ∎

2.3. Stability and accuracy of the Hood–Taylor elements. In this section, we shall establish the following inf-sup condition: that for $\kappa \ge 1$,

$$\sup_{0 \ne \breve{\mathbf{v}}_h \in \breve{\mathbf{V}}_h} \frac{(\nabla \cdot \breve{\mathbf{v}}_h, \breve{p}_h)_0}{|\breve{\mathbf{v}}_h|_1} \gtrsim \|\breve{p}_h\|_0 \quad \forall \breve{p}_h \in \breve{S}_h. \tag{2.26}$$

For this purpose, we shall follow the argument introduced by Stenberg [22], the so-called macroelement technique.

The macroelement technique introduced is a very useful technique establishing the stability of the finite element pairs for the Stokes equation. The main ingredients in this technique consist of the macroelement $\breve{\mathcal{M}}_h = \{\breve{M}_k\}_k$ decomposition of the domain $\breve{\Omega}$, the local inf-sup condition for the finite element pairs in each macroelement $\breve{M}_k$ that belongs to $\breve{\mathcal{M}}_h$ using the dimensional argument, and combining them to establish the global inf-sup condition. Although it is standard in the classical finite elements, our finite element spaces are not standard in that we are considering "toroid" finite element discretization. Therefore, we shall need to rephrase the classical theory [22] in our framework. However, it turns out that a minimal modification and proof will be needed to transfer the classical theories to fit in our framework.

To define a macroelement in our framework, we begin to use the classical definition of the macroelement used by Stenberg [22] for the domain Ω, which shall be denoted by $\mathcal{M}_h = \{M_k\}_k$. We then define our macroelement by rotating it around the z-axis, namely,

$$\breve{\mathcal{M}}_h = \{\breve{M}_k = (r\cos\theta, r\sin\theta, z) : (r,z) \in M_k\}_k. \tag{2.27}$$

Our choice for M_k is the patch associated with each vertex ν_k of the triangulation $\mathcal{T}_h$. This choice of macroelement partition can be shown to satisfy the following:

1. each $M \in \mathcal{M}_h$ belongs to an equivalence class of macroelements;
2. the number of macroelement classes is finite;
3. each $\tau \in \mathcal{T}_h$ belongs to at most a finite number N of macroelements $M \in \mathcal{M}_h$ with N independent of h.

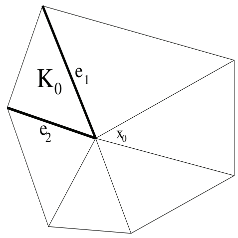

FIG. 2.1. *Macroelement with an internal node x_0 and triangle K_0 with two edges e_i with $i = 1, 2$.*

We shall transfer definitions given by Stenberg [22] to our framework $\breve{\mathcal{M}}_h$ by saying that, for example, if $\mathcal{E}_M$ is the equivalence class, then $\mathcal{E}_{\breve{M}}$ is the corresponding equivalence class for "toroid" finite element cases. We shall further introduce the space denoted by $N_{\breve{M}}$ for each $\breve{M} \in \breve{\mathcal{M}}_h$ as follows:

$$N_{\breve{M}} = \left\{ \breve{q} \in \breve{S}_{\breve{M}} : \int_{\breve{M}} \breve{q}\, \nabla \cdot \breve{\mathbf{v}} = 0, \quad \breve{\mathbf{v}} \in \breve{\mathbf{V}}_{0,\breve{M}} \right\}, \tag{2.28}$$

where $\breve{S}_{\breve{M}} = \{\breve{p} \in \breve{L}^2(\breve{M}) : p|_\tau \in \mathcal{P}^\kappa(\tau)\}$ and $\breve{\mathbf{V}}_{0,\breve{M}} = \breve{\mathbf{V}}_h \cap \breve{\mathbf{H}}_0^1(\breve{M})$. We shall denote $\breve{S}_{0,\breve{M}} = \breve{S}_{\breve{M}} \cap L_0^2(\breve{M})$ and two dimensional representation of $\breve{S}_{\breve{M}}$ and $\breve{S}_{0,\breve{M}}$ by $\overline{S}_M$ and $\overline{S}_{0,M}$, respectively. The following lemma is crucial and the idea of the proof comes from Boffi [5].

LEMMA 2.8. *Suppose that every element $\tau \in \mathcal{T}_h$ has at least one vertex in the interior of Ω. Define $\mathcal{M}_h$ by grouping together, for each internal vertex x_0, those elements which touch x_0. Then for every $M \in \mathcal{M}_h$, the null space $N_{\breve{M}}$ is one dimensional, consisting of those functions which are constant on $\breve{M}$.*

Proof. Consider a generic macroelement $M \in \mathcal{M}_h$. Let $K_0 \in \mathcal{T}_h$ be a triangle of M and denote by x_0 the internal vertex of K_0 which also belongs to the other elements of M. There are two edges e_i with $i = 1, 2$ of K_0 meeting at x_0 (see Figure 2.1 for an illustration). Due to the fact that x_0 is internal, none of the edges e_i lie on the boundary $\partial\Omega$. Let $p \in S_M$ be given and suppose that

$$\int_{\breve{M}} \breve{p} \nabla \cdot \breve{\mathbf{v}} d\mathbf{x} = 0 \quad \forall \breve{\mathbf{v}} \in \breve{\mathbf{V}}_{0,\breve{M}}. \tag{2.29}$$

We shall prove that $\nabla\breve{p}$ vanishes on $\breve{K}_0$, thus obtaining the anticipated result by virtue of the fact that $\breve{K}_0$ is arbitrary and $\breve{p}$ is continuous. First, we concentrate our attention on the edge e_1 and consider two triangles K_0 and K_1 that belong to $\mathcal{T}_h$ which share the edge e_1 in common. We now define $\breve{\mathbf{v}}$ in the following way: for each $i = 0, 1$,

$$\breve{\mathbf{v}}|_{\breve{K}_i} = \left(\breve{\lambda}_i \breve{\mu}_i \breve{\partial_{e_1}} p\, \partial_{e_1} r \cos\theta, \breve{\lambda}_i \breve{\mu}_i \breve{\partial_{e_1}} p\, \partial_{e_1} r \sin\theta, \breve{\lambda}_i \breve{\mu}_i \breve{\partial_{e_1}} p\, \partial_{e_1} z \right)^t,$$

where λ_i and μ_i are functions for the two edges of K_i that are different from e_1 chosen to take positive values in the interior of K_i. We then define $\breve{\mathbf{v}}|_K = (0, 0, 0)^t$ on any other element K that belongs to $\breve{M}$. It is then easy to validate that $\breve{\mathbf{v}} \in \breve{\mathbf{V}}_h$ and it is of degree $k + 1$ while $\breve{p}$ is of degree k. We note that $\partial_{e_1} p = \partial_r p\, \partial_{e_1} r + \partial_z p\, \partial_{e_1} z$ and the following identity holds true: if $\int_{\breve{M}} \nabla\breve{p} \cdot \breve{\mathbf{v}} d\mathbf{x} = 0$, then

$$\int_M (\partial_r p, \partial_z p) \cdot (\lambda_i \mu_i \partial_{e_1} p\, \partial_{e_1} r, \lambda_i \mu_i \partial_{e_1} p\, \partial_{e_1} z)\; r dr dz = \sum_i \int_{K_i} \lambda_i \mu_i \left|\partial_{e_1} p\right|^2 r dr dz = 0.$$

Therefore, $\partial_{e_1} p = 0$, in particular on K_0. On the other hand, we can show that $\partial_{e_2} p = 0$ on K_0 as well by a similar argument. Therefore, we show that p is constant on K_0 since e_1 and e_2 are linearly independent. Now, by the arbitrariness of the choice of K_0, we can establish that p must be a constant on M. This completes the proof. ∎

A consequence of the aforementioned Lemma 2.8 is the following theorem.

THEOREM 2.9. *Let $\mathcal{E}_{\breve{M}}$ be a class of equivalent macroelements. For $\breve{M} \in \mathcal{E}_{\breve{M}}$, it holds true that*

$$\sup_{0\neq\breve{\mathbf{v}}\in\breve{\mathbf{V}}_{0,\breve{M}}} \frac{(\nabla\cdot\breve{\mathbf{v}},\breve{p})_{0,\breve{M}}}{|\breve{\mathbf{v}}|_{1,\breve{M}}} \gtrsim \|\breve{p}\|_{0,\breve{M}} \quad \forall \breve{p}\in\breve{S}_{0,\breve{M}}. \tag{2.30}$$

Proof. The proof can be done basically from the idea presented by Stenberg [22]. Therefore, we provide a sketch of the proof. We can assume $v_\theta = 0$ and note that $(\nabla\cdot\breve{\mathbf{v}},\breve{p})_{0.\breve{M}} \cong (v_r, \partial_r p)_{0,1,M} + (v_z, \partial_z p)_{0,1,M}$ and $|\breve{\mathbf{v}}|^2_{1,\breve{M}} \cong |v_r|^2_{V^1_1(M)} + |v_z|^2_{1,1,M}$. We shall need to deal with the equivalence classes of macroelements that touch the z-axis and that do not touch the z-axis separately. For the macroelement M, which is away from the z-axis, we can show that $|\breve{\mathbf{v}}|^2_{1,\breve{M}} \cong r_{max}(M)\left[|v_r|^2_{1,0,M} + |v_z|^2_{1,0,M}\right]$, which results from the inequality $\int_M v_r^2/r^2 drdz \le C|v_r|^2_{1,0,M}$ for C being independent of M (see Corollary 4.1 in [19]). Similarly, we have

$$(\nabla\cdot\breve{\mathbf{v}},\breve{p})_{0.\breve{M}} \cong r_{max}(M)\left[(v_r,\partial_r p)_{0,M} + (v_z,\partial_z p)_{0,M}\right].$$

This will lead to the inequality (2.30) for any such M following the arguments in [22]. On the other hand, for the macroelement M that touches the z-axis, the inequality (2.30) can be shown to be valid from the argument of Lemma 3.1 in [22] in the framework of the weighed Sobolev spaces. This completes the proof. ∎

At this point, these local inf-sup conditions need to be combined to make the global inf-sup condition, which will be possible from the following lemma. The proof can be found in the appendix.

LEMMA 2.10. *There exists an interpolation operator $\breve{\mathcal{I}}_h : \breve{\mathbf{H}}^1_0(\breve{\Omega}) \mapsto \breve{\mathbf{V}}_h$ such that $\forall\breve{\mathbf{u}}\in\breve{\mathbf{H}}^1_0(\breve{\Omega})$,*

$$\int_{\breve{\Omega}} \nabla\cdot(\breve{\mathbf{u}} - \breve{\mathcal{I}}_h\breve{\mathbf{u}})d\mathbf{x} = 0 \quad \textit{and} \quad \|\breve{\mathcal{I}}_h\breve{\mathbf{u}}\|_1 \lesssim \|\breve{\mathbf{u}}\|_1. \tag{2.31}$$

We shall introduce a subspace $\breve{C}_h \subset \breve{S}_h$ defined by

$$\breve{C}_h = \left\{\breve{\mu}\in\breve{L}^2_0(\breve{\Omega}) : \breve{\mu}_{|\breve{M}} \in \breve{\mathcal{P}}^0(\breve{M}) \quad \forall\breve{M}\in\breve{\mathcal{M}}_h\right\} \tag{2.32}$$

and denote by $\breve{\gamma}_h$ the L^2 projection from $\breve{S}_h$ onto the space $\breve{C}_h$. We are in position to provide the main theorem in this section.

THEOREM 2.11. *Under the assumption proposed for the macroelement partition $\breve{\mathcal{M}}_h$, the inf-sup condition* (2.26) *for Hood–Taylor finite elements for any degree $\kappa \ge 1$ holds true.*

Proof. Using the argument provided in Lemma 3.2 in Stenberg [22], we can show that for every $\breve{p}_h \in \breve{S}_h$, there is a $\breve{\mathbf{v}}_h \in \breve{\mathbf{V}}_h$ such that

$$(\nabla\cdot\breve{\mathbf{v}}_h,\breve{p}_h) \gtrsim \|(\boldsymbol{\delta}-\breve{\gamma}_h)\breve{p}_h\|^2_0 \quad \text{and} \quad |\breve{\mathbf{v}}_h|_1 \lesssim \|(\boldsymbol{\delta}-\breve{\gamma}_h)\breve{p}_h\|_0. \tag{2.33}$$

We now show that for every $\breve{p}_h \in \breve{S}_h$, there is $\breve{\mathbf{w}}_h \in \breve{\mathbf{V}}_h$ such that

$$(2.34)\qquad (\nabla \cdot \breve{\mathbf{w}}_h, \breve{\gamma}_h \breve{p}_h) = \|\breve{\gamma}_h \breve{p}_h\|_0^2 \quad \text{and} \quad |\breve{\mathbf{w}}_h|_1 \lesssim \|\breve{\gamma}_h \breve{p}_h\|_0.$$

The existence of such a function $\breve{\mathbf{w}}_h$ can be obtained from Theorem 2.1. Namely, for $\breve{\mu}_h = \breve{\gamma}_h \breve{p}_h \in \breve{C}_h$, we choose $\breve{\mathbf{w}} \in \breve{\mathbf{H}}_0^1(\breve{\Omega})$ such that $\nabla \cdot \breve{\mathbf{w}} = \breve{\mu}_h$ and $\|\breve{\mathbf{w}}\|_1 \lesssim \|\breve{\mu}_h\|_0$. At this point, we can now apply the same argument provided in Theorem 3.1 in Stenberg [22] to conclude our proof. ∎

From the inf-sup condition, we obtain the following optimal error estimate.

THEOREM 2.12. *Suppose that $\breve{\mathbf{u}}$ and $\breve{p}$ smooth enough and $\breve{\mathbf{V}}_h \times \breve{S}_h$ are of piecewise continuous polynomials of degree $\kappa + 1$ and κ, respectively, with $\kappa \geq 1$. Then we have the following optimal error estimates:*

$$(2.35)\qquad \|\breve{\mathbf{u}} - \breve{\mathbf{u}}_h\|_{1,\breve{\Omega}} + \|\breve{p} - \breve{p}_h\|_{0,\breve{\Omega}} \lesssim h^{\kappa+1} \left(\|\breve{\mathbf{u}}\|_{\kappa+2} + \|\breve{p}\|_{\kappa+1}\right).$$

3. Preconditioning.

In this section, we shall discuss the fast solver for the axisymmetric Stokes equation (2.10). For the solution, we shall consider the preconditioned minimum residual iterative method by Rusten and Winther [20]. The classical preconditioner for the generalized Stokes equation can be found at Bramble and Pasciak [7]. We begin by casting the Stokes equation (2.2) as the following operator equation: $\breve{\mathcal{S}}_h \breve{U}_h = \breve{F}_h$, where $\breve{\mathcal{S}}_h = (S_{ij})_{ij=1,2}$ with $S_{11} = \breve{\mathcal{A}}_h$, $S_{12} = \breve{\mathcal{B}}_h^*$, $S_{21} = \breve{\mathcal{B}}_h$ and $S_{22} = 0$, $\breve{U}_h = (\breve{\mathbf{u}}_h, \breve{p}_h)^t$, and $\breve{F} = (\breve{\mathbf{f}}, 0)^t$. Here $\breve{\mathcal{A}}_h$ is the discrete Laplace operator and $\breve{\mathcal{B}}_h$ is the discrete divergence operator, respectively. It is easy to show that the application of the minimum residual iterative methods for the axisymmetric formulation of the Stokes equation and the full three dimensional Stokes restricted to the axisymmetric solution spaces are equivalent. Therefore, it is enough to precondition the operator $\breve{\mathcal{S}}_h$ applied to the axisymmetric solution space. The actual implementation can be followed by interpreting the operator in terms of basis and the underlying inner product. It is well known that the operator $\breve{\mathcal{S}}_h$ is spectrally equivalent to the following block diagonal operator: $\mathcal{D}_h = (D_{ij})_{i,j=1,2}$ with $D_{11} = \breve{\mathcal{A}}_h$, $D_{22} = \breve{\mathcal{B}}_h \breve{\mathcal{A}}_h^{-1} \breve{\mathcal{B}}_h$, and $D_{ij} = 0$ for $i \neq j$. Therefore, we shall have to introduce spectrally equivalent operators for both $\breve{\mathcal{A}}$ and the Schur complement operator $\breve{\mathcal{B}}\breve{\mathcal{A}}^{-1}\breve{\mathcal{B}}$. The main task in this section is to show that the operator $\breve{\mathcal{A}}^{-1}$ is spectrally equivalent to the standard multigrid backslash cycle methods based on Gauss–Seidel smoothing. On the other hand, the spectrally equivalent operator for the Schur complement operator given by $\breve{\mathcal{B}}\breve{\mathcal{A}}^{-1}\breve{\mathcal{B}}$ can be seen from the following simple identity:

$$(3.1)\qquad (\breve{\mathcal{B}}\breve{\mathcal{A}}^{-1}\breve{\mathcal{B}}\breve{p}, \breve{p})_0 = \sup_{\breve{\mathbf{v}}_h \in \breve{\mathbf{V}}_h} \frac{(\breve{p}, \nabla \cdot \breve{\mathbf{v}})_0^2}{(\breve{\mathbf{v}}, \breve{\mathbf{v}})_1}.$$

It is clear that the identity indicates the Schur complement operator is spectrally equivalent to the identity operator. Therefore, the associated matrix should be given in terms of the mass matrix with respect to a $(\cdot,\cdot)_{0,1}$ inner product. We observe that our viewpoint here can provide very simple construction of the preconditioner based on the classical theories.

3.1. Vector Laplacian.

In this section, we shall show that the standard multigrid backslash cycle will be spectrally equivalent to the vector Laplace operator $\breve{\mathcal{A}}_h$ in the (1,1) block of the operator $\breve{\mathcal{S}}_h$. Throughout this section, for simplicity of presentation, boldface shall not be used either to denote functions or function spaces, and we omit the subscript h. The main subject in this section is to solve the following discrete weak formulation by the use of the multigrid methods: find $\breve{u} \in \breve{V}$ such that $a(\breve{u}, \breve{v}) = \langle \breve{f}, \breve{v} \rangle \ \forall \breve{v} \in \breve{V}$. Note that $\breve{V}$ is a real Hilbert space with an inner product $a(\cdot,\cdot)$ and (energy) norm $\|\cdot\| = a(\cdot,\cdot)^{1/2}$.

3.2. MSSC for axisymmetric vector Laplace equations. The construction of the general subspace correction methods is based on the space decomposition that the space $\breve{V}$ is decomposed into a number of subspaces $\breve{V}_k$, with $k = 1, \ldots, J$ and the introduction of the local subspace solver in each subspace.

Assumption A0. There are closed subspaces $\{\breve{V}_k\}_{k=1}^L$ such that $\breve{V} = \sum_{k=1}^L \breve{V}_k$. For each subspace $\breve{V}_k$, we define the orthogonal projection $\breve{P}_k : \breve{H}_0^1(\breve{\Omega}) \mapsto \breve{V}_k$ with respect to an $a(\cdot,\cdot)$ inner product by

$$a(\breve{P}_k \breve{v}, \breve{v}_k) = a(\breve{v}, \breve{v}_k) \quad \forall \breve{v} \in \breve{V}, \breve{v}_k \in \breve{V}_k. \tag{3.2}$$

Note that the bilinear form $a(\cdot,\cdot)$ is coercive on $\breve{V}_k$ for each $k = 1, \ldots, J$; therefore, the operator $\breve{P}_k$ is well-defined. These shall be used as our local subspace solvers. The method of subspace corrections can be found in [25].

Algorithm 1 (MSSC). *Let $\breve{u}^0 \in \breve{V}$ be given.*

for $\ell = 1, 2, \ldots$
  $\breve{u}_0^{\ell-1} = \breve{u}^{\ell-1}$
  for $i = 1, \ldots, L$
    Let $\breve{e}_i \in \breve{V}_i$ solve

$$a(\breve{e}_i, \breve{v}_i) = \breve{f}(\breve{v}_i) - a(\breve{u}_{i-1}^{\ell-1}, \breve{v}_i) \quad \forall \breve{v}_i \in \breve{V}_i \tag{3.3}$$

    $\breve{u}_i^{\ell-1} = \breve{u}_{i-1}^{\ell-1} + \breve{e}_i$
  endfor
  $\breve{u}^\ell = \breve{u}_L^{\ell-1}$
endfor

The following result can be easily established from Assumption A0 [14, 17].

Theorem 3.1. *Under Assumption A0, the estimate of the energy norm for the error transfer operator $\breve{E}_L = (I - \breve{P}_L) \cdots (I - \breve{P}_1)$ can be established as follows:*

$$\|\breve{E}_L\|^2 = 1 - K^{-1}, \ \text{with } K = \sup_{\|\breve{v}\|=1} \inf_{\sum_{k=1}^L \breve{v}_k = \breve{v}} \sum_{k=1}^L \|\breve{P}_k \sum_{j \geq k} \breve{v}_j\|^2 \ \text{and } \breve{v}_k \in \breve{V}_k \ \forall k. \tag{3.4}$$

3.2.1. Multilevel finite element spaces. Throughout this section, we assume that we have a nested sequence of triangulations $\breve{\mathcal{T}}_k = \{\breve{\tau}_k^i\}$, $1 \leq k \leq L$ of $\breve{\Omega}$ with characteristic mesh size h_k proportional to γ^{2k} with $\gamma \in (0,1)$. The nested sequence of triangles will be assumed to be formed in such a way that the refined triangle is obtained by connecting the midpoints of the coarse triangles for simplicity. Let $\breve{\mathcal{T}}_h = \breve{\mathcal{T}}_L$ and $\breve{V}_k$ denote the spaces corresponding $\breve{V}_h$ defined on the triangulations $\breve{\mathcal{T}}_k$. In the rest of this section, to simplify notation, we shall omit the subscript h when referring to a fixed finest triangulation, namely, $\breve{V} = \breve{V}_h = \breve{V}_L$. For the discussions that follow, we describe the space decompositions and subspace corrections, and we let $\{x_k^\ell\}_\ell$ be a set of nodes for the triangulation $\breve{\mathcal{T}}_k$. For $1 \leq k \leq L$, we set $\breve{V}_k^\ell = \operatorname{span}\{\breve{\phi}_k^\ell\}$, where $\{\breve{\phi}_k^\ell\}_k^\ell$ is the basis for the space $\breve{V}_k$:

$$\breve{V} = \sum_{k=1}^L \breve{V}_k = \sum_{k=1}^L \sum_{\ell=1}^{N_k} \breve{V}_k^\ell, \tag{3.5}$$

where N_k is the number of nodes for the triangulation $\breve{\mathcal{T}}_k$. We introduce the L^2 projection, $\breve{Q}_k : L^2(\breve{\Omega}) \mapsto \breve{V}_k$, for each $k = 1, \ldots, J$. The corresponding L^2 and H^1 projections onto $\breve{V}_k^\ell$ will be denoted by $\breve{Q}_k^\ell$ and $\breve{P}_k^\ell$, respectively. The multigrid algorithm shall be constructed in particular in terms of the algorithm MSSC with the local exact solver, $\breve{P}_k^\ell$ in each subspace $\breve{V}_k^\ell$. By the aforementioned decompositions, we can show that the algorithm is the multigrid method with the smoother being the Gauss–Seidel with pointwise smoothing. We note that the spaces $\{V_k\}_{k=1,\ldots,L}$ are nested, namely, $\breve{V}_1 \subset \cdots \subset \breve{V}_k \subset \cdots \subset \breve{V}_L$. Finally, from Lemmas A.2, A.6, and A.7, we can establish that there exists an interpolation operators $\breve{\mathbf{\Pi}}_h : \breve{H}_0^1(\breve{\Omega}) \mapsto \breve{V}$ such that

$$\breve{\mathbf{\Pi}}_h \breve{v} = \breve{v} \quad \forall \breve{v} \in \breve{V} \quad \text{and} \quad \|\breve{v} - \breve{\mathbf{\Pi}}_h \breve{v}\|_0 \lesssim h|\breve{v}|_1 \quad \forall \breve{v} \in \breve{H}_0^1(\breve{\Omega}). \tag{3.6}$$

We shall now prove some instrumental results for the multigrid convergence analysis.

LEMMA 3.2. *The following inverse inequality holds true:*

$$|\breve{v}|_1 \lesssim h^{-1}\|\breve{v}\|_0 \quad \forall \breve{v} \in \breve{V}. \tag{3.7}$$

Proof. Note that $\breve{v}$ takes the form that $\breve{v} = (v_r \cos\theta - v_\theta \sin\theta, v_r \sin\theta + v_\theta \cos\theta, v_z)^t$, with $v_r, v_\theta \in V_{1,0}^1, v_z \in H_{1,0}^1$, and the following inequality holds true:

$$|\breve{v}|_{1,\breve{\tau}}^2 \lesssim |v_r|_{1,1,\tau}^2 + \|v_r\|_{0,-1,\tau}^2 + |v_\theta|_{1,1,\tau}^2 + \|v_\theta\|_{0,-1,\tau}^2 + |v_z|_{1,1,\tau}^2. \tag{3.8}$$

Thus, it is enough to show that $|v_r|_{1,1,\tau} \lesssim h^{-1}|v_r|_{0,1,\tau}$ and $\|v_r\|_{0,-1,\tau} \lesssim h^{-1}\|v_r\|_{0,1,\tau}$. Let $\widehat{\tau}$ denote the reference triangle of τ, and let $\widehat{v}_r$ denote the function v_r defined on the reference triangle. We observe that if τ intersects the z-axis, by the scaling argument and the norm equivalence (see Belhachmi, Bernardi, and Deparis [2] for details, and see our appendix as well), we have $|v_r|_{1,1,\tau}^2 \lesssim h^{-2}\|v_r\|_{0,1,\tau}^2$. On a toroid type $\breve{\tau}$, we let $r_{\min} = \min_r\{(r,z) \in \tau\}$ and $r_{\max} = \max_r\{(r,z) \in \tau\}$; we then apply the following inequality:

$$|v_r|_{1,1,\tau}^2 \lesssim r_{min}|v_r|_{1,0,\tau}^2 \lesssim r_{min}h^{-2}\|v_r\|_{0,0,\tau}^2 \lesssim h^{-2}\|v_r\|_{0,1,\tau}^2.$$

On the other hand, we observe that no matter where the location of τ is, it holds true that $|v_r|_{0,-1,\tau}^2 \lesssim r_{max}^{-2}|v_r|_{0,1,\tau}^2 \lesssim h^{-2}|v_r|_{0,1,\tau}^2$. This completes the proof. ∎

From the approximation property of the interpolation operator as well as the inverse inequality from Lemma 3.2, we can obtain the stability result for the L^2 projection operator $\breve{Q}_k$ as follows.

LEMMA 3.3. *For each $1 \le k \le L$, the operator $\breve{Q}_k$ is stable with respect to a $|\cdot|_\alpha$ seminorm for $\alpha \in [0,1]$, namely,*

$$|\breve{Q}_k \breve{v}|_\alpha \lesssim |\breve{v}|_\alpha \quad \forall \breve{v} \in \breve{V}.$$

Additionally, the following strengthened Cauchy–Schwarz inequality can be shown from the inverse inequality together with the standard trace estimate [8].

LEMMA 3.4. *Assume that $1 \le i \le j \le L$. The following holds true:*

$$(\nabla \breve{u}_j : \nabla \breve{v}_i) \lesssim \gamma^{j-i} h_j^{-1} \|\breve{u}_j\|_0 \|\nabla \breve{v}_i\|_0 \quad \forall \breve{u}_j \in \breve{V}_j, \breve{v}_i \in \breve{V}_i. \tag{3.9}$$

LEMMA 3.5. *Let $\breve{v} \in \breve{V}_h$ with $\overline{v} = (v_r, v_\theta, v_z)^t$ and v_r, v_θ, v_z belong to $P_h^{\kappa+1,0}(\tau)$ with $\breve{\tau} \in \breve{\mathcal{T}}_k$ and* $\operatorname{diam}\tau = h_\tau$. *Then for each $k = 1, \ldots, L$, the following relation holds true:*

$$\|\breve{v}\|_{0,\breve{\tau}}^2 \cong r_{max(\tau)} h_\tau^2 \sum_{i=1}^{(\kappa+2)(\kappa+3)/2} \left(v_r^2(x_i) + v_\theta^2(x_i) + v_z(x_i)^2\right), \tag{3.10}$$

where $\{x_i\}$ is the set of nodes where the degrees of freedom of $\breve{v}$ are defined on $\breve{\tau}$ and $r_{\max(\breve{\tau})} = \max\{\sqrt{x^2+y^2} : (x,y,z) \in \breve{\tau}\}$.

Proof. If $\tau \in \mathcal{T}$ intersects the z-axis, it is affine equivalent to a standard triangle with an edge on the z-axis or the standard triangle with single vertex on the z-axis. Let $\hat{\tau}$ be the reference triangle. Since all norms are equivalent norms on the finite dimensional space span$\{\hat{\phi}_i|_{\hat{\tau}}\}$ and using the scaling argument, for a function $w \in P_h^{\kappa+1}(\tau)$, we have that for τ intersecting the z-axis,

$$\|w\|_{0,1,\tau}^2 \cong h^3\|\hat{w}\|_{0,1,\hat{\tau}}^2 \cong h^3\|\hat{w}\|_{\infty,\hat{\tau}}^2 \cong h^3 \sum \hat{w}^2(\hat{x}_i). \tag{3.11}$$

For a toroid type element, $\breve{\tau}$, similarly to the aforementioned case, we obtain that

$$\|w\|_{0,1,\tau}^2 \cong r_{max}(\tau)\|w\|_{0,\tau}^2 \cong r_{max}(\tau)h^2\|\hat{w}\|_{0,\hat{\tau}}^2 \cong r_{max}(\tau)h^2 \sum \hat{w}^2(\hat{x}_i).$$

This completes the proof. ∎

Lemma 3.6. *For any $\breve{v} \in \breve{V}_h$, the following holds true:*

$$\sum_{k=1}^{L} \|(\breve{Q}_k - \breve{Q}_{k-1})\breve{v}\|_1^2 \lesssim \|\breve{v}\|_1^2. \tag{3.12}$$

Proof. Let $\widetilde{Q}_k = \breve{Q}_k - \breve{Q}_{k-1}$ and $\bar{v}_i = (\breve{P}_i - \breve{P}_{i-1})\breve{v}$, where for each $i = 1, \ldots, J$,

$$(\breve{P}_i\breve{v}, \breve{v}_i)_1 = (\breve{v}, \breve{v}_i)_1 \quad \forall \breve{v}_i \in \breve{V}_i.$$

We note that the following inequalities hold true:

$$|\widetilde{Q}_k\bar{v}_i|_1^2 \lesssim h_k^{-2\alpha}|\widetilde{Q}_k\bar{v}_i|_{1-\alpha}^2 \lesssim h_k^{-2\alpha}|\bar{v}_i|_{1-\alpha}^2 \lesssim h_k^{-2\alpha}h_i^{2\alpha}|\bar{v}_i|_1^2.$$

Set $i \wedge j = \min\{i,j\}$. We then obtain that

$$\begin{aligned}
\sum_{k=1}^{L} |(\breve{Q}_k - \breve{Q}_{k-1})\breve{v}|_1^2 &= \sum_{k=1}^{L}\sum_{i,j=k}^{L} (\widetilde{Q}_k\bar{v}_i, \widetilde{Q}_k\bar{v}_j)_1 = \sum_{i,j=1}^{L}\sum_{k=1}^{i\wedge j} (\widetilde{Q}_k\bar{v}_i, \widetilde{Q}_k\bar{v}_j)_1 \\
&\lesssim \sum_{i,j=1}^{L}\sum_{k=1}^{i\wedge j} h_k^{-2\alpha}h_i^{\alpha}h_j^{\alpha}|\bar{v}_i|_1|\bar{v}_j|_1 \lesssim \sum_{i,j=1}^{L} h_{i\wedge j}^{-2\alpha}h_i^{\alpha}h_j^{\alpha}|\bar{v}_i|_1|\bar{v}_j|_1 \\
&= \sum_{i,j=1}^{L} \gamma^{2\alpha|i-j|}|\bar{v}_i|_1|\bar{v}_j|_1 \lesssim \sum_{k=1}^{L} |\bar{v}_k|_1^2 = |\breve{v}|_1^2.
\end{aligned}$$

This completes the proof. ∎

Based on established aforementioned technical lemmas, it is then easy to conclude the following theorem from the argument in [13].

Theorem 3.7. *The following estimate holds true: $\|\breve{E}_L\|^2 = 1 - K^{-1}$ with*

$$K \lesssim \sup_{\|\breve{v}\|=1 \in \breve{V}} \inf_{\sum_{k=1}^{L}\sum_{i=1}^{N_k} \breve{v}_k^i = \breve{v}} \sum_{k=1}^{L}\sum_{i=1}^{N_k} \left\| \breve{P}_k^i \sum_{(\ell,j)\geq(k,i)} \breve{v}_\ell^j \right\|^2 \leq \delta < 1, \tag{3.13}$$

where $\breve{v}_k^i \in \breve{V}_k^i$, for each k and i and δ is independent of h.

4. Numerical experiments. In this subsection, we shall present sample numerical results that confirm some of our theoretical analysis. We consider the following axisymmetric Stokes equation on the domain $\Omega = [0,1] \times [0,1]$:

$$\begin{aligned} -\partial_r^2 u_r - r^{-1}\partial_r u_r - \partial_z^2 u_r + r^{-2}u_r + \partial_r p &= f_r, \\ -\partial_r^2 u_z - r^{-1}\partial_r u_z - \partial_z^2 u_z + \partial_z p &= f_z, \\ -\partial_r u_r - r^{-1}u_r - \partial_z u_z &= 0, \end{aligned}$$

where $f_r = r^3 \sin z$ and $f_z = 8r^2 \cos z - 16 \cos z$. The choice of this force function $\overline{\mathbf{f}} = (f_r, f_z)^t$ corresponds to the pair of analytic solutions $\overline{\mathbf{u}} = (u_r, u_\theta, u_z)^t$ with $u_r = r^3 \sin z$, $u_\theta = 0$, and $u_z = 4r^2 \cos z$, and $p = 4r^2 \sin z$. We shall denote the discrete axisymmetric Stokes equation by the following operator equation:

$$\mathsf{S}_h U_h = F_h, \tag{4.1}$$

where $\mathsf{S}_h = (S_{ij})_{ij=1,2}$ with $S_{11} = \mathsf{A}_h$, $S_{12} = \mathsf{B}_h^*$, $S_{21} = \mathsf{B}_h$ and $S_{22} = 0$, $U_h = (\overline{\mathbf{u}}_h, p_h)^t$, and $F_h = (\overline{\mathbf{f}}_h, 0)^t$. For the solution to the aforementioned equation, we have tested the preconditioned minimum residual iteration with the block diagonal preconditioner $\mathsf{D}_h = (D_{ij})_{i,j=1,2}$ given by $D_{11} = \mathsf{C}_h$ and $D_{22} = \mathsf{M}_h$, where C_h^{-1} is one iteration of Hypre algebraic multigrid solver, which can be assumed to be spectrally equivalent to the vector Laplacian A_h since we have shown that the standard multigrid with point Gauss–Seidel smoothing can be an effective solver for A_h, and M_h is the mass matrix with respect to $(\cdot,\cdot)_{0,1}$ inner product. Table 4.1 demonstrates that the proposed block diagonal matrix is spectrally equivalent to the axisymmetric Stokes operator. The stopping criteria used is $\|\widehat{\mathsf{S}}^{-1} R^k\| / \|\widehat{\mathsf{S}}^{-1} F_h\| < 10^{-6}$, where R^k is the kth residual. We have also investigated the error behavior, which excellently agrees with the theoretical analysis. Table 4.2 shows the convergence order for the $\breve{L}^2$ error for the pressure and the $\breve{\mathbf{H}}^1$ error for the velocity is 2 for $\kappa = 1$.

TABLE 4.1
The number of iterations of the preconditioned MINRES for Hood–Taylor discrete axisymmetric Stokes with $\kappa = 1$ as a function of the mesh size. The computation is done using a Dual-Core Intel Xeon Processor with a processor speed of 2.66 GHz.

Mesh size	# of iterations	CPU (in seconds)
$1/2^3$	56	7.18E-02
$1/2^4$	58	1.52E-01
$1/2^5$	58	5.77E-01
$1/2^6$	58	2.30E+00
$1/2^7$	58	1.10E+01

TABLE 4.2
The $\breve{L}^2$ error for the velocity and the pressure and the $\breve{\mathbf{H}}^1$ error for the velocity as a function of the mesh size for Hood–Taylor discrete axisymmetric Stokes with $\kappa = 1$.

Mesh size	$\|\breve{\mathbf{u}} - \breve{\mathbf{u}}_h\|_0$		$\|\breve{\mathbf{u}} - \breve{\mathbf{u}}_h\|_1$		$\|\breve{p} - \breve{p}_h\|_0$	
$1/2^3$	0.13E-03	x	0.80E-02	x	0.73E-02	x
$1/2^4$	0.17E-04	2.95	0.20E-02	1.98	0.17E-02	2.0
$1/2^5$	0.21E-05	2.98	0.51E-03	1.99	0.43E-03	2.0
$1/2^6$	0.27E-06	2.99	0.12E-03	1.99	0.10E-03	2.0

5. Conclusion. This paper provides the stability analysis of the Hood–Taylor finite elements for the axisymmetric formulation of the Stokes equation for any degree $\kappa \geq 1$. The newly developed framework can effectively use the standard theoretical results. An illustration is given for the development and analysis for the preconditioner of the axisymmetric Stokes operator. Sample numerical results are provided to confirm our theoretical results as well.

Appendix. This section provides proofs of various results stated in the main text. We first give the following lemma regarding the scaling in the weighted norms on a triangle intersecting the z-axis. We require that if the intersection set is a point (resp., an edge), then the reference triangle intersects the z-axis on a point (resp., an edge). Namely, there are two reference triangles corresponding to different cases.

LEMMA A.1. *Let E be either a triangle or an edge in the triangulation intersecting but not contained in the z-axis. Let $h = \text{diam}(\text{E})$. Then, for a function v on E,*

$$|v|_{\ell,1,E} \cong h^{1/2+\kappa-\ell}|\hat{v}|_{\ell,1,\hat{E}}, \quad \ell = 0, 1,$$

where $\kappa = 1$ on a triangle and $\kappa = 1/2$ on an edge.

Proof. Let $\{\lambda_i\}_{i=0,1,2}$ be the barycentric coordinates associated with the vertex x_i of the triangle including E, $i = 1, 2, 3$. Assume the triangle has an edge on the z-axis and x_1 is away from the z-axis. Define $r_1 = r(x_1)$. Then, if E is a triangle,

$$|v|^2_{0,1,E} = \int_E v^2 r dr dz = r_1 \int_E v^2 \lambda_1 dr dz = |E| r_1 \int_{\hat{E}} \hat{v}^2 \hat{\lambda}_1 d\hat{r} d\hat{z} \cong h^3 |\hat{v}|^2_{0,1,\hat{E}}.$$

If E is an edge,

$$|v|^2_{0,1,E} = \int_E v^2 r ds = r_1 \int_E v^2 \lambda_1 ds = |E| r_1 \int_{\hat{E}} \hat{v}^2 \hat{\lambda}_1 d\hat{s} \cong h^2 |\hat{v}|^2_{0,1,\hat{E}}.$$

The case $\ell = 1$ and the case when the triangle containing E intersects the z-axis on a vertex follow a similar calculation using the barycentric coordinates. ∎

A.1. Proof of Theorems 2.4 and 2.5. In this section, we shall provide technical results for proofs of Theorems 2.4 and 2.5.

LEMMA A.2. *For a compact set $K \subset \bar{\Omega}$, let $h_K = \text{diam}(K) < 1$. Suppose K is star-shaped with respect to a ball of radius $\delta\, h_K$. If $K \cap \{r = 0\} \neq \emptyset$, then there exists $p \in \mathcal{P}^\kappa(K)$ such that for all $\kappa \geq \ell$,*

$$|v - p|_{\ell,1,K} \leq C h_K^{\kappa-\ell} |v|_{\kappa,1,K} \quad \forall v \in H^\kappa_1(K), \tag{A.1}$$

where the constant C depends on δ, but not on v or h_K.

Proof. The proof for $\kappa = 0, 1$ and $\ell = 0, 1$ can be found in [2, 18]. For $\ell \geq 2$, $\kappa \geq 2$, the proof can be done following a similar process. For example, see Lemma 7 in [2]. ∎

Note that based on the standard approximation results (see [8]) in the usual Sobolev spaces, the above lemma also holds for K, on which $r \geq C h_K$.

LEMMA A.3. *Let ϕ be a usual Lagrange basis function on the triangle τ. Then the following holds true:*

- *If $\tau \cap \{r = 0\} \neq \emptyset$, $|\phi|_{\ell,1,\tau} \lesssim h^{3/2-\ell}$, $\ell = 0, 1$.*
- *If $\tau \cap \{r = 0\} = \emptyset$, $|\phi|_{\ell,1,\tau} \lesssim r_{max}(\tau)^{1/2} h^{1-\ell}$, $\ell = 0, 1$.*

Proof. These inequalities can be proven by the usual scaling argument. If τ does not intersect the z-axis, we have $|\phi|_{\ell,1,\tau} \leq r_{max}(\tau)^{1/2} |\phi|_{\ell,\tau} \lesssim r_{max}(\tau)^{1/2} h^{1-\ell}$.

Now, if τ intersects the z-axis, then $|\phi|_{\ell,1,\tau} \lesssim h^{1/2}|\phi|_{\ell,\tau} \lesssim h^{3/2-\ell}$. This completes the proof. ▯

LEMMA A.4. *Let ϕ be a usual Lagrange basis function that vanishes on the z-axis in case $\tau \cap \{r = 0\} \neq \emptyset$. Then the following holds true: for $\ell = 0, 1$,*

- *if $\tau \cap \{r = 0\} \neq \emptyset$, then*

$$\|\phi\|_{0,-1,\tau} \lesssim h^{1/2} \quad \text{and} \quad |\phi|_{\ell,1,\tau} \lesssim h^{3/2-\ell}; \tag{A.2}$$

- *if $\tau \cap \{r = 0\} = \emptyset$, then*

$$\|\phi\|_{0,-1,\tau} \lesssim r_{min}(\tau)^{-1/2} h \quad \text{and} \quad |\phi|_{\ell,1,\tau} \lesssim r_{max}(\tau)^{1/2} h^{1-\ell}. \tag{A.3}$$

Proof. When τ does not intersect the z-axis, r is bounded below, and thus inequalities in (A.3) can be obtained by the standard argument. We shall prove the case when τ intersects the z-axis. Note that the estimate on $|\cdot|_{\ell,1,\tau}$, the second inequality of (A.2), can be obtained using the norm equivalence and the usual scaling arguments. For the proof of the first inequality in (A.2), we assume $\tau \cap \{r = 0\}$ is an edge. Then, since ϕ is a polynomial that vanishes on the z-axis, we can write $\phi = rq$, where q is a polynomial. Namely, r is a factor of ϕ. Therefore, $\|\phi\|_{0,-1,\tau} < \infty$ and $\|\cdot\|_{0,-1,\tau}$ defines a norm for ϕ. Let $\{\lambda_i\}_{i=0,1,2}$ be the barycentric coordinates associated with the vertex x_i of τ, $i = 1, 2, 3$. Let x_1 be the vertex away from the z-axis and let r_1 be the distance from x_1 to the z-axis. It is easy to verify that the distance $r(x)$ to the z-axis for any point x in τ can be written as $r(x) = r_1\lambda_1(x)$. Note $r_1 \cong h$. Using the norm equivalence on finite dimensional spaces and the second estimate in (A.2), we observe

$$\begin{aligned}\|\phi\|_{0,-1,\tau} &= \|\phi r^{-1/2}\|_{0,\tau} = r_1^{-1/2}\|\phi\lambda_1^{-1/2}\|_{0,\tau} = |\tau|^{1/2} r_1^{-1/2}\|\widehat{\phi\lambda_1^{-1/2}}\|_{0,\widehat{\tau}} \\ &\lesssim |\tau|^{1/2} r_1^{-1/2}\|\widehat{\phi}\widehat{\lambda}_1^{1/2}\|_{0,\widehat{\tau}} \lesssim r_1^{-1}\|\phi r_1^{1/2}\lambda_1^{1/2}\|_{0,\tau} = r_1^{-1}\|\phi\|_{0,1,\tau} \lesssim h^{1/2}.\end{aligned}$$

We now assume that $\tau \cap \{r = 0\}$ is a vertex and that x_1 and x_2 are the vertices of τ away from the z-axis and $x_3 = (0, z_3)$. Note that ϕ is assumed to vanish on x_3, and hence vanishes on at least one of the two edges e_1 and e_2 of τ containing x_3. These edges can be written as $a_\ell r + z - z_3 = 0$, $\ell = 1, 2$, respectively. Therefore, $\phi = (a_\ell r + z - z_3)q$ for some ℓ, with q being a polynomial. Without loss of generality, we let $\ell = 1$. We consider $\widetilde{\tau} \in \tau$, a triangle enclosed by the edges e_ℓ with $\ell = 1, 2$ and the vertical line $r = \epsilon$, for ϵ small. We note that

$$\begin{aligned}\int_{\widetilde{\tau}} \phi^2 r^{-1} dr dz &= \left| \int_0^\epsilon \int_{z_3-a_1 r}^{z_3-a_2 r} (a_1 r + z - z_3)^2 q^2 r^{-1} dz dr \right| \\ &\le \|(a_1 r + z - z_3)q^2\|_{\infty,\widetilde{\tau}} \left| \int_0^\epsilon \int_{z_3-a_1 r}^{z_3-a_2 r} (a_1 r + z - z_3) r^{-1} dz dr \right| \\ &\lesssim \left| \int_0^\epsilon (a_1 - a_2)^2 r dr \right| < \infty.\end{aligned}$$

Thus, $\|\cdot\|_{0,-1,\tau}$ defines a norm for ϕ. Let r_1 and r_2 be the distances from x_1 and x_2 to the z-axis, respectively. Note that for any $x \in \tau$, $r(x) = r_1\lambda_1(x) + r_2\lambda_2(x)$, and $r_1 \cong r_2 \cong h$ based on the shape regularity of the mesh. Therefore, using the norm equivalence on finite dimensional spaces and the second estimate in (A.2),

$$\|\phi\|_{0,-1,\tau} = \left\|\phi(r_1\lambda_1 + r_2\lambda_2)^{-1/2}\right\|_{0,\tau} \lesssim |\tau|^{1/2}\left\|\widehat{\phi}(r_1\widehat{\lambda}_1 + r_2\widehat{\lambda}_2)^{-1/2}\right\|_{0,\widehat{\tau}}$$

$$\lesssim |\tau|^{1/2} h^{-1/2} \left\| \widehat{\phi}(\widehat{\lambda}_1 + \widehat{\lambda}_2)^{-1/2} \right\|_{0,\widehat{\tau}} \lesssim |\tau|^{1/2} h^{-1/2} \left\| \widehat{\phi}(\widehat{\lambda}_1 + \widehat{\lambda}_2)^{1/2} \right\|_{0,\widehat{\tau}}$$
$$\lesssim h^{-1} \left\| \phi(r_1\lambda_1 + r_2\lambda_2)^{1/2} \right\|_{0,\tau} \lesssim h^{-1} \|\phi r^{1/2}\|_{0,\tau} \lesssim h^{-1} \|\phi\|_{0,1,\tau} \lesssim h^{1/2},$$

which completes the proof of this lemma. ∎

We shall now consider the norm of the operator π_i defined in section 2.2.1.

Lemma A.5. *The following estimates hold true:*

- *If $\pi_i v$ is defined by the triangle $\tau(x_i)$ intersecting the z-axis,*

$$\|\pi_i v\|_{\infty,\tau(x_i)} \lesssim h^{-3/2} \|v\|_{0,1,\tau(x_i)}.$$

- *If $\pi_i v$ is defined by the edge $e(x_i)$ intersecting the z-axis,*

$$\|\pi_i v\|_{\infty,e(x_i)} \lesssim h^{-1/2} |v|_{1,1,\tau_i} + h^{-3/2} \|v\|_{0,1,\tau_i}.$$

- *If $\pi_i v$ is defined by the triangle $\tau(x_i)$ not intersecting the z-axis,*

$$\|\pi_i v\|_{\infty,\tau(x_i)} \lesssim r_{min}(\tau(x_i))^{-1/2} h^{-1} \|v\|_{0,1,\tau(x_i)}.$$

- *If $\pi_i v$ is defined by the edge $e(x_i)$ not intersecting the z-axis,*

$$\|\pi_i v\|_{\infty,e(x_i)} \lesssim r_{min}(\tau_i)^{-1/2} (|v|_{1,1,\tau_i} + h^{-1} \|v\|_{0,1,\tau_i}).$$

Proof. Let E be either the triangle $\tau(x_i)$ or the edge $e(x_i)$, where π_i is defined. We have the following inequality from the definition of π_i:

$$\|\pi_i v\|_{0,1,E} \leq \|v\|_{0,1,E}. \tag{A.4}$$

We first prove the case that π_i is defined in terms of the triangle $\tau(x_i)$. If $\tau(x_i)$ does not intersect the z-axis, using the norm equivalence, we have

$$\|\pi_i v\|_{\infty,\tau(x_i)} = \|\hat{\pi}_i \hat{v}\|_{\infty,\hat{\tau}(x_i)} \lesssim \|\hat{\pi}_i \hat{v}\|_{0,\hat{\tau}(x_i)} \lesssim h^{-1} \|\pi_i v\|_{0,\tau(x_i)}$$
$$\lesssim r_{min}(\tau(x_i))^{-1/2} h^{-1} \|\pi_i v\|_{0,1,\tau(x_i)} \lesssim r_{min}(\tau(x_i))^{-1/2} h^{-1} \|v\|_{0,1,\tau(x_i)}.$$

Now for $\tau(x_i)$ that touches the z-axis,

$$\|\pi_i v\|_{\infty,\tau(x_i)} = \|\hat{\pi}_i \hat{v}\|_{\infty,\hat{\tau}(x_i)} \lesssim \|\hat{\pi}_i \hat{v}\|_{0,1,\hat{\tau}(x_i)}$$
$$\lesssim h^{-3/2} \|\pi_i v\|_{0,1,\tau(x_i)} \lesssim h^{-3/2} \|v\|_{0,1,\tau(x_i)}.$$

We consider the case that π_i is defined in terms of the edge $e(x_i)$. If $e(x_i)$ does not intersect the z-axis, then by using the standard trace estimates and the fact that $r_{max}(\tau_i)/r_{min}(\tau_i) \lesssim 1$, we have

$$\|\pi_i v\|_{\infty,e(x_i)} = \|\hat{\pi}_i \hat{v}\|_{\infty,\hat{e}(x_i)} \leq \|\hat{\pi}_i \hat{v}\|_{0,\hat{e}(x_i)} \lesssim h^{-1/2} \|\pi_i v\|_{0,e(x_i)}$$
$$\lesssim r_{min}(\tau_i)^{-1/2} h^{-1/2} \|\pi_i v\|_{0,1,e(x_i)} \lesssim r_{min}(\tau_i)^{-1/2} h^{-1/2} \|v\|_{0,1,e(x_i)}$$
$$\lesssim r_{max}(\tau_i)^{1/2} r_{min}(\tau_i)^{-1/2} h^{-1/2} \|v\|_{0,e(x_i)}$$
$$\lesssim h^{-1/2} (h^{1/2} |v|_{1,\tau_i} + h^{-1/2} \|v\|_{0,\tau_i})$$
$$\lesssim r_{min}(\tau_i)^{-1/2} (|v|_{1,1,\tau_i} + h^{-1} \|v\|_{0,1,\tau_i}).$$

We now assume that $e(x_i)$ intersects the z-axis and the triangle τ_i intersects the z-axis at an edge. Then, from a weighted trace theorem in [10], we obtain

$$\begin{aligned}\|\pi_i v\|_{\infty,e(x_i)} &= \|\hat{\pi}_i\hat{v}\|_{\infty,\hat{e}(x_i)} \lesssim \|\hat{\pi}_i\hat{v}\|_{0,1,\hat{e}(x_i)} \lesssim h^{-1}\|\pi_i v\|_{0,1,e(x_i)} \lesssim h^{-1}\|v\|_{0,1,e(x_i)} \\ &\lesssim h^{-1}\left(h^{1/2}|v|_{1,1,\tau_i} + h^{-1/2}\|v\|_{0,1,\tau_i}\right) \lesssim h^{-1/2}|v|_{1,1,\tau_i} + h^{-3/2}\|v\|_{0,1,\tau_i}.\end{aligned}$$

If $e(x_i)$ intersects the z-axis, and the triangle τ_i intersects the z-axis at a point, then we first observe that from the definition of $\hat{\pi}_i$ and Hölder's inequality,

$$\|\hat{\pi}_i\hat{v}\|^2_{0,1,\hat{e}(x_i)} \leq \|\hat{\pi}_i\hat{v}\|_{0,0,\hat{e}(x_i)}\|\hat{v}\|_{0,2,\hat{e}(x_i)}. \tag{A.5}$$

Then, using the norm equivalence for functions in the finite dimensional space,

$$\|\hat{\pi}_i\hat{v}\|_{0,1,\hat{e}(x_i)} \lesssim \|\hat{v}\|_{0,2,\hat{e}(x_i)}.$$

Therefore, by a weighted trace theorem in [10], we obtain

$$\begin{aligned}\|\pi_i v\|_{\infty,e(x_i)} &= \|\hat{\pi}_i\hat{v}\|_{\infty,\hat{e}(x_i)} \lesssim \|\hat{\pi}_i\hat{v}\|_{0,1,\hat{e}(x_i)} \lesssim \|\hat{v}\|_{0,2,\hat{e}(x_i)} \lesssim h^{-3/2}\|v\|_{0,2,e(x_i)} \\ &\lesssim h^{-3/2}(h|v|_{1,1,\tau_i} + \|v\|_{0,1,\tau_i}) \lesssim h^{-1/2}|v|_{1,1,\tau_i} + h^{-3/2}\|v\|_{0,1,\tau_i}.\end{aligned}$$

This completes the proof. ∎

The following lemma proves Theorem 2.4.

LEMMA A.6. *For a given $\tau \in \mathcal{T}_h$, let U_τ be the union of triangles that intersect τ. Then for $\ell = 0, 1$, the interpolation operator $\Pi_h^+ : H_1^1(\Omega) \mapsto P_h^\kappa(\Omega)$ satisfies*

$$|\Pi_h^+ v|_{\ell,1,\tau} \lesssim h^{1-\ell}|v|_{1,1,U_\tau} + h^{-\ell}\|v\|_{0,1,U_\tau} \quad \forall v \in H_1^1(\Omega). \tag{A.6}$$

Proof. Let x_i be a node in $\bar{\tau}$. First, we assume τ does not intersect the z-axis. Therefore, based on the usual estimates on the trace, Lemmas A.5 and A.4, if π_i is defined in terms of the edge $e(x_i)$, then we have

$$\begin{aligned}&|\pi_i v(x_i)\phi|_{\ell,1,\tau} \lesssim \|\pi_i v\|_{\infty,e(x_i)}|\phi|_{\ell,1,\tau} \lesssim r_{max}(\tau)^{1/2}h^{1-\ell}\|\pi_i v\|_{\infty,e(x_i)} \\ &\lesssim r_{max}(\tau)^{1/2}h^{1-\ell}r_{min}(\tau_i)^{-1/2}(|v|_{1,1,\tau_i} + h^{-1}\|v\|_{0,1,\tau_i}) \lesssim h^{1-\ell}|v|_{1,1,\tau_i} + h^{-\ell}\|v\|_{0,1,\tau_i}.\end{aligned}$$

If π_i is defined in terms of the triangle $\tau(x_i)$ that does not intersect the z-axis, then

$$\begin{aligned}|\pi_i v(x_i)\phi|_{\ell,1,\tau} &\lesssim \|\pi_i v\|_{\infty,\tau(x_i)}|\phi|_{\ell,1,\tau} \lesssim r_{max}(\tau)^{1/2}h^{1-\ell}\|\pi_i v\|_{\infty,\tau(x_i)} \\ &\lesssim r_{max}(\tau)^{1/2}h^{1-\ell}r_{min}(\tau(x_i))^{-1/2}h^{-1}\|v\|_{0,1,\tau(x_i)} \lesssim h^{-\ell}\|v\|_{0,1,\tau(x_i)}.\end{aligned}$$

In the case that $\tau(x_i)$ intersects the z-axis, a similar argument shows that

$$|\pi_i v(x_i)\phi|_{\ell,1,\tau} \lesssim h^{-\ell}\|v\|_{0,1,\tau(x_i)}.$$

Second, we assume that τ intersects the z-axis. If π_i is defined in terms of the edge $e(x_i)$ that does not intersect the z-axis, then by Lemma A.5 we have

$$\begin{aligned}&|\pi_i v(x_i)\phi|_{\ell,1,\tau} \lesssim \|\pi_i v\|_{\infty,e(x_i)}|\phi|_{\ell,1,\tau} \lesssim h^{3/2-\ell}\|\pi_i v\|_{\infty,e(x_i)} \\ &\lesssim h^{3/2-\ell}r_{min}(\tau_i)^{-1/2}(|v|_{1,1,\tau_i} + h^{-1}\|v\|_{0,1,\tau_i}) \lesssim h^{1-\ell}|v|_{1,1,\tau_i} + h^{-\ell}\|v\|_{0,1,\tau_i}.\end{aligned}$$

If $e(x_i)$ intersects the z-axis, then, using Lemma A.5, we have

$$\begin{aligned}|\pi_i v(x_i)\phi|_{\ell,1,\tau} &\lesssim \|\pi_i v\|_{\infty,e(x_i)}|\phi|_{\ell,1,\tau} \lesssim h^{3/2-\ell}\|\pi_i v\|_{\infty,e(x_i)} \\ &\lesssim h^{3/2-\ell}(h^{-1/2}|v|_{1,1,\tau_i} + h^{-3/2}\|v\|_{0,1,\tau_i}) \lesssim h^{1-\ell}|v|_{1,1,\tau_i} + h^{-\ell}\|v\|_{0,1,\tau_i}.\end{aligned}$$

If π_i is defined in terms of the $\tau(x_i)$ that does not intersect the z-axis, by Lemma A.5,

$$\begin{aligned}|\pi_i v(x_i)\phi|_{\ell,1,\tau} &\lesssim \|\pi_i v\|_{\infty,\tau(x_i)}|\phi_i|_{\ell,1,\tau} \lesssim h^{3/2-\ell}\|\pi_i v\|_{\infty,\tau(x_i)}\\ &\lesssim h^{3/2-\ell} r_{min}(\tau(x_i))^{-1/2} h^{-1}\|v\|_{0,1,\tau(x_i)} \lesssim h^{-\ell}\|v\|_{0,1,\tau(x_i)}.\end{aligned}$$

If $\tau(x_i)$ intersects the z-axis, by Lemma A.5, we have

$$\begin{aligned}|\pi_i v(x_i)\phi|_{\ell,1,\tau} &\lesssim \|\pi_i v\|_{\infty,\tau(x_i)}|\phi|_{\ell,1,\tau} \lesssim h^{3/2-\ell}\|\pi_i v\|_{\infty,\tau(x_i)}\\ &\lesssim h^{3/2-\ell}h^{-3/2}\|v\|_{0,1,\tau(x_i)} \lesssim h^{-\ell}\|v\|_{0,1,\tau(x_i)}.\end{aligned}$$

Therefore,

$$|\Pi_h^+ v|^2_{\ell,1,\tau} \lesssim \sum_{i,\ x_i\in\bar\tau} |\pi_i v(x_i)\phi_i|^2_{\ell,1,\tau} \lesssim h^{2-2\ell}|v|^2_{1,1,U_\tau} + h^{-2\ell}\|v\|^2_{0,1,U_\tau}.$$

This completes the proof. ∎

We note that from the above estimates and Lemma A.2, for any $p \in \mathcal{P}^\kappa(U_\tau)$,

$$\begin{aligned}|v-\Pi_h^+ v|_{\ell,1,\tau} &\le |v-p|_{\ell,1,\tau} + |\Pi_h^+(v-p)|_{\ell,1,\tau}\\ &\lesssim |v-p|_{\ell,1,\tau} + h^{1-\ell}|v-p|_{1,1,U_\tau} + h^{-\ell}\|v-p\|_{0,1,U_\tau}\\ &\lesssim h^{\kappa+1-\ell}\|v\|_{\kappa+1,1,U_\tau},\end{aligned}$$

which completes the proof of Theorem 2.4. The following lemma gives the stability result in Theorem 2.5.

LEMMA A.7. *For a given $\tau \in \mathcal{T}_h$. Let U_τ be the union of triangles that intersect τ. The interpolation operator $\Pi_h^- : V_1^1(\Omega) \mapsto P_h^\kappa(\Omega)$ satisfies, for $\ell = 0, 1$,*

$$\|\Pi_h^- v\|_{0,-1,\tau} \lesssim \|v\|_{V_1^1(U_\tau)} \quad \forall v \in V_1^1(\Omega) \tag{A.7}$$

$$|\Pi_h^- v|_{\ell,1,\tau} \lesssim h^{1-\ell}|v|_{1,1,U_\tau} + h^{-\ell}\|v\|_{0,1,U_\tau} \quad \forall v \in V_1^1(\Omega). \tag{A.8}$$

In addition, for any $v \in H_1^2(\Omega)$, the following holds true:

- *if $\tau \cap \{r = 0\} \neq \emptyset$, then*

$$|\Pi_h^- v|_{\ell,\tau} \lesssim h^{-1/2-\ell}\|v\|_{0,1,U_\tau} + h^{1/2-\ell}|v|_{1,1,U_\tau} + h^{3/2-\ell}|v|_{2,1,U_\tau}; \tag{A.9}$$

- *if $\tau \cap \{r = 0\} = \emptyset$, then*

$$|\Pi_h^- v|_{\ell,\tau} \lesssim r_{max}^{-1/2}(\tau)h^{-\ell}\left(\|v\|_{0,1,U_\tau} + h|v|_{1,1,U_\tau}\right). \tag{A.10}$$

Proof. We recall that

$$\Pi_h^- v = \sum_{i,x_i\notin\{r=0\}} \Pi_h^+ v(x_i)\phi_i + \sum_{i,x_i\in\{r=0\}} \pi_{i,r} v(x_i)\phi_i. \tag{A.11}$$

Then, for any $v \in V_1^1(\Omega)$, it is clear that $\pi_{i,r}v(x_i) = 0$ for any node x_i lying on the z-axis. Therefore, we estimate only for the node x_i away form the z-axis. By Lemma A.4 and following the same process as in Lemma A.6, for different $\pi_i v$ defined by a triangle or an edge, we have $\|\pi_i v(x_i)\phi_i\|_{0,-1,\tau} \lesssim \|v\|_{0,-1,U_\tau}$ and $\|\pi_i v(x_i)\phi_i\|_{0,-1,\tau} \lesssim |v|_{1,1,U_\tau} + \|v\|_{0,-1,U_\tau}$, respectively. Then the first desired inequality is obtained by summing up the above estimates over the triangle. Similarly, we have for $\ell = 0, 1$, $|\Pi_h^- v|_{\ell,1,\tau} \lesssim h^{1-\ell}|v|_{1,1,U_\tau} + h^{-\ell}\|v\|_{0,1,U_\tau}$, for both τ intersecting the z-axis and τ away from the z-axis, which proves (A.8).

We now prove inequalities (A.9) and (A.10) of Lemma A.7. We first note the following result in [19]. Let P be a bounded domain in the rz-plane, $r \geq 0$, intersecting the z-axis; then $\|v\|_{1,P} \leq C\|v\|_{2,1,P}\ \forall v \in H^2_1(P)$. Therefore, v has a trace on each edge e_i belonging to the z-axis in the L^2 sense. Then, for a triangle τ with $x_i \in \{r=0\}$ as a node, by Lemma A.4 and the trace estimate for functions in $H^1(\Omega)$, we have

$$\begin{aligned}|\pi_{i,r}v(x_i)\phi_i|_{\ell,\tau} &\leq \|\pi_{i,r}v\|_{\infty,e(x_i)}|\phi_i|_{\ell,\tau} \lesssim h^{1-\ell}\|\pi_{i,r}v\|_{\infty,e(x_i)} \lesssim h^{1-\ell}\|\widehat{\pi}_{i,r}\widehat{v}\|_{\infty,\widehat{e}}\\ &\lesssim h^{1-\ell}\|\widehat{\pi}_{i,r}\widehat{v}\|_{0,\widehat{e}} \lesssim h^{1-\ell}\|\widehat{v}\|_{0,\widehat{e}} \lesssim h^{1-\ell}\|\hat{v}\|_{1,\hat{\tau}} \lesssim h^{1-\ell}\|\hat{v}\|_{2,1,\hat{\tau}}\\ &\lesssim h^{1-\ell}(h^{-3/2}\|v\|_{0,1,\tau_i} + h^{-1/2}|v|_{1,1,\tau_i} + h^{1/2}|v|_{2,1,\tau_i})\\ &\lesssim h^{-1/2-\ell}\|v\|_{0,1,\tau_i} + h^{1/2-\ell}|v|_{1,1,\tau_i} + h^{3/2-\ell}|v|_{2,1,\tau_i}.\end{aligned}$$

For any node x_i away from the z-axis, using the same process as in Lemma A.6, for $\pi_i v$ defined by a triangle, we have $|\pi_i v(x_i)\phi_i|_{\ell,\tau} \lesssim h^{1-\ell}h^{-1}r_{max}(\tau)^{-1/2}\|v\|_{0,1,\tau(x_i)}$. For $\pi_i v$ defined by an edge, we have $|\pi_i v(x_i)\phi_i|_{\ell,\tau} \lesssim r_{max}(\tau)^{-1/2}h^{-\ell}(\|v\|_{0,1,\tau_i} + h|v|_{1,1,\tau_i})$. Then, (A.9) and (A.10) follow by adding these estimates in the corresponding triangle. □

The approximation result in Theorem 2.5 is proved as follows.

LEMMA A.8. *For $\kappa \geq 0$, suppose the finite element space contains continuous piecewise polynomials of degree κ. Then,*

$$\|v - \Pi_h^- v\|_{V_1^1} \lesssim h^\kappa(\|v\|_{\kappa+1,1,\Omega} + \|v\|_{0,-1,\Omega}) \quad \forall v \in H_1^{\kappa+1}(\Omega) \cap L^2_{-1}(\Omega). \tag{A.12}$$

Proof. For $\kappa = 0$, by Lemma A.7, we have $\|\Pi_h^- v\|_{0,-1,\Omega} \lesssim |v|_{1,1,\Omega} + \|v\|_{0,-1,\Omega}$. Note that $\|v - \Pi_h^- v\|_{1,1,\Omega} \leq \|v - \Pi_h^+ v\|_{1,1,\Omega} + \|\Pi_h^+ v - \Pi_h^- v\|_{1,1,\Omega}$. Since the first term has the desired approximation rate, we concentrate on the second term. Note that the second term vanishes on triangles away from the z-axis. Thus,

$$\|\Pi_h^+ v - \Pi_h^- v\|_{1,1,\Omega} \lesssim \sum_{i, x_i \in \{r=0\}} h^{1/2}\|\pi_i v\|_{\infty,E_i},$$

where E_i is either the triangle or edge that defines $\pi_i v$. By Lemma A.5, we have

$$\|\pi_i v\|_{\infty,E_i} \lesssim h^{-3/2}\|v\|_{0,1,U_{E_i}} + h^{-1/2}|v|_{1,1,U_{E_i}} \lesssim h^{-1/2}\|v\|_{0,-1,U_{E_i}} + h^{-1/2}|v|_{1,1,U_{E_i}},$$

where U_{E_i} is the union of adjacent triangles of E_i.

Therefore, for $\kappa = 0$, $\|v - \Pi_h^- v\|_{1,1,\Omega} + \|v - \Pi_h^- v\|_{0,-1,\Omega} \lesssim |v|_{1,1,\Omega} + \|v\|_{0,-1,\Omega}$. For $\kappa \geq 1\ \forall v \in H_1^{\kappa+1}(\Omega) \cap L^2_{-1}(\Omega)$, we first have (see [19]) $\|v\|_{1,\Omega} \lesssim \|v\|_{\kappa+1,1,\Omega}$.

Then, if τ intersects the z-axis, recall $\Pi^- p = p$ for any $p \in P_h^\kappa$, by Lemma A.7 and Corollary 4.1 in [19],

$$\begin{aligned}\|v - \Pi_h^- v\|_{0,-1,\tau} &\lesssim h^{1/2}\|\hat{v} - \hat{\Pi}_h^- \hat{v}\|_{0,-1,\hat{\tau}} \lesssim h^{1/2}\|\hat{v} - \hat{\Pi}_h^- \hat{v}\|_{1,\hat{\tau}}\\ &\lesssim h^{1/2}(\|\hat{v} - \hat{p}\|_{1,\hat{\tau}} + \|\hat{\Pi}_h^- \hat{v} - \hat{\Pi}_h^- \hat{p}\|_{1,\hat{\tau}})\\ &\lesssim h^{1/2}(\|\hat{v} - \hat{p}\|_{1,\hat{\tau}} + \|\hat{v} - \hat{p}\|_{2,1,\hat{U}_\tau}) \lesssim h^{1/2}\|\hat{v} - \hat{p}\|_{\kappa+1,1,\hat{U}_\tau}.\end{aligned}$$

Since by Lemma A.2, $\inf_{\hat{p}\in\mathcal{P}^\kappa}\|\hat{v} - \hat{p}\|_{\kappa+1,1,\hat{U}_\tau} \lesssim |\hat{v}|_{\kappa+1,1,\hat{U}_\tau}$, we have $\|v - \Pi_h^- v\|_{0,-1,\tau} \lesssim h^{1/2}h^{\kappa-1/2}|v|_{\kappa+1,1,U_\tau} \lesssim h^\kappa\|v\|_{\kappa+1,1,U_\tau}$. If τ is away from the z-axis, note $\Pi_h^- = \Pi_h^+$ and

$$\begin{aligned}\|v - \Pi_h^- v\|_{0,-1,\tau} &\lesssim r_{min}(\tau)^{-1}\|v - \Pi_h^+ v\|_{0,1,\tau}\\ &\lesssim r_{min}(\tau)^{-1}h^{\kappa+1}\|v\|_{\kappa+1,1,U_\tau} \lesssim h^\kappa\|v\|_{\kappa+1,1,U_\tau}.\end{aligned}$$

Now for $\|v - \Pi_h^- v\|_{1,1,\tau}$, by Lemmas A.7 and A.2, we have

$$\|v - \Pi_h^- v\|_{1,1,\tau} \le \|v - p\|_{1,1,\tau} + \|\Pi_h^- p - \Pi_h^- v\|_{1,1,\tau} \lesssim h^\kappa \|v\|_{\kappa+1,1,U_\tau}.$$

Summing up the estimates over all the triangles proves this lemma. ◻

A.2. Construction of the interpolation operator $\breve{\mathcal{I}}_h$. In this section, we shall construct the interpolation operator $\breve{\mathcal{I}}_h : \breve{\mathbf{H}}_0^1(\breve{\Omega}) \mapsto \breve{\mathbf{V}}_h$ that satisfies

$$\text{(A.13)} \qquad \int_{\breve{\Omega}} \nabla \cdot (\breve{\mathbf{u}} - \breve{\mathcal{I}}_h \breve{\mathbf{u}}) d\mathbf{x} = 0 \quad \text{and} \quad \|\breve{\mathcal{I}}_h \breve{\mathbf{u}}\|_1 \lesssim \|\breve{\mathbf{u}}\|_1 \quad \forall \breve{\mathbf{u}} \in \breve{\mathbf{H}}_0^1(\breve{\Omega}).$$

For this purpose, we shall need some preliminary results. The first result is a simple consequence of the work by Copeland, Gopalaknishnan, and Pasciak in [10].

PROPOSITION A.9. *Let $\tau \subset \Omega$ be a triangle with diameter h and let $e \subset \tau$ be an edge. For any $v \in \mathcal{C}^\infty(\bar{\tau})$, the following estimates hold true.*

Case 1. *If $\bar{\tau} \cap \{r = 0\} \ne \emptyset$ and $\bar{e} \cap \{r = 0\} \ne \emptyset$ but $e \notin \{r = 0\}$, then*

$$\text{(A.14)} \qquad \int_e vrde \lesssim h^{1/2} \|v\|_{0,1,\tau} + h^{3/2} |v|_{1,1,\tau}.$$

Case 2. *If $\bar{\tau} \cap \{r = 0\} = \emptyset$, then*

$$\text{(A.15)} \qquad \int_e vrde \lesssim \left(\int_e rde \right)^{1/2} \left(h^{-1/2} \|v\|_{0,1,\tau} + h^{1/2} |v|_{1,1,\tau} \right).$$

From Proposition A.9, we obtain the following estimates.

PROPOSITION A.10. *Let $\tau \subset \Omega$ be a triangle with diameter h, and let e be an edge of τ that does not belong to the z-axis. Suppose that $\phi \in \mathcal{P}^2(\tau)$ is the basis function that corresponds to the midpoint of the edge e. Then,*

$$\frac{\int_e vrde}{\int_e rde} \|\phi\|_{\ell,1,\tau} \lesssim h^{-\ell} \|v\|_{0,1,\tau} + h^{1-\ell} |v|_{1,1,\tau} \ \textit{and}\ \frac{\int_e vrde}{\int_e rde} \|\phi\|_{0,-1,\tau} \lesssim \|v\|_{0,-1,\tau} + |v|_{1,1,\tau}.$$

Proof. Suppose $\bar{e} \cap \{r = 0\} \ne \emptyset$. Then, it holds true that $h^2 \lesssim \int_e rde \lesssim h^2$. Then, the first inequality is the direct consequence of Proposition A.9 and Lemma A.3. The second inequality can be similarly obtained by using Proposition A.9 and Lemma A.4. Namely, we have $\frac{\int_e vrde}{\int_e rde} \|\phi\|_{0,-1,\tau} \lesssim h^{-3/2} \int_e vrde \lesssim h^{-1} \|v\|_{0,1,\tau} + |v|_{1,1,\tau} \lesssim \|v\|_{0,-1,\tau} + |v|_{1,1,\tau}$. In case $\bar{e} \cap \{r = 0\} = \emptyset$, the estimates can be obtained using the fact that $\int_e rde \approx r_{min}(\tau) h$. This completes the proof. ◻

We are in a position to construct the operator $\breve{\mathcal{I}}_h$. For a triangulation $\mathcal{T}_h$ of Ω, we collect all the edges $\{e_k : e_k \subset \tau \in \mathcal{T}_h\}$ that do not belong to the z-axis. We now consider the standard basis function $\{\phi_k\}_k$ that is associated with the midpoint of the edge e_k for each k. The main idea is based on the fact [2] that for each k, there exist generic constants ρ_k and ν_k for which the following identity holds true: $\int_{e_k} \phi_k rde = (\int_{e_k} rde) \rho_k = \nu_k$, from which we can define the modified basis function ψ_k by $\psi_k = \frac{\phi_k}{\nu_k}$. Then it is clear by definition that $\int_{e_j} \psi_k rde = \delta_{jk}$. Furthermore, ψ_k takes zero at all vertices of the triangle in $\mathcal{T}_h$. Three steps will be taken to construct $\breve{\mathcal{I}}_h$. First, we define an operator $\Pi_h^0 : H_1^1(\Omega) \mapsto P_h^2(\Omega)$ by $\Pi_h^0 u = \sum_k (\int_{e_k} urde) \psi_k \, \forall u \in H_1^1(\Omega)$. Second, we define $\breve{\mathbf{\Pi}}_h^0 : \breve{\mathbf{H}}_0^1(\breve{\Omega}) \mapsto \breve{\mathbf{V}}_h$ by

$$\text{(A.16)} \qquad \breve{\mathbf{\Pi}}_h^0 \breve{\mathbf{u}} = \left(\Pi_h^0 u_r \cos\theta - \Pi_h^0 u_\theta \sin\theta, \Pi_h^0 u_r \sin\theta + \Pi_h^0 u_\theta \cos\theta, \Pi_h^0 u_z \right)^t,$$

where $\breve{\mathbf{u}} = (u_r \cos\theta - u_\theta \sin\theta, u_r \sin\theta + u_\theta \cos\theta, u_z)^t$. Finally, we now define the operator $\breve{\mathcal{I}}_h : \breve{\mathbf{H}}_0^1(\breve{\Omega}) \mapsto \breve{\mathbf{V}}_h$ by $\breve{\mathcal{I}}_h = \breve{\mathbf{\Pi}}_h + \breve{\mathbf{\Pi}}_h^0(\boldsymbol{\delta} - \breve{\mathbf{\Pi}}_h)$, where $\boldsymbol{\delta}$ is the identity operator and the operator $\breve{\mathbf{\Pi}}_h$ is the interpolation operator introduced in section 2.2. Note that by construction, it is easy to establish that for $\breve{\mu}_h \in \breve{C}_h$, $(\nabla \cdot \breve{\mathcal{I}}_h \breve{\mathbf{u}}, \breve{\mu}_h)_0 = (\nabla \cdot \breve{\mathbf{u}}, \breve{\mu}_h)_0$. Therefore, it is clear to see that

$$\int_{\breve{\Omega}} \nabla \cdot (\breve{\mathbf{u}} - \breve{\mathcal{I}}_h \breve{\mathbf{u}}) d\mathbf{x} = 0 \quad \forall \breve{\mathbf{u}} \in \breve{\mathbf{H}}_0^1(\breve{\Omega}).$$

Furthermore, we can establish that $\breve{\mathcal{I}}_h$ is stable. Note that the operator $\breve{\mathcal{I}}_h$ preserves polynomials of degree two, namely, $\breve{\mathcal{I}}_h \breve{\boldsymbol{p}} = \breve{\boldsymbol{p}}\ \forall \breve{\boldsymbol{p}} \in \breve{P}_h^2$. Now, from Proposition A.10, it holds true that for $m = 0, 1$, $\|\breve{\mathcal{I}}_h \breve{\mathbf{u}}\|_m \lesssim h^{-m} \|\breve{\mathbf{u}}\|_0 + h^{-m+1} |\breve{\mathbf{u}}|_1$. Therefore, we obtain the following relation: $\|\breve{\mathcal{I}}_h \breve{\mathbf{u}} - \breve{\mathbf{u}}\|_1 \le \|\breve{\mathbf{u}} - \breve{\boldsymbol{p}}\|_1 + \|\breve{\mathcal{I}}_h(\breve{\mathbf{u}} - \breve{\boldsymbol{p}})\|_1 \lesssim \|\breve{\mathbf{u}}\|_1$, from which we can complete the proof that $\|\breve{\mathcal{I}}_h \breve{\mathbf{u}}\|_1 \le \|\breve{\mathbf{u}}\|_1 + \|\breve{\mathbf{u}} - \breve{\mathcal{I}}_h \breve{\mathbf{u}}\|_1 \lesssim \|\breve{\mathbf{u}}\|_1$.

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REFERENCES

[1] R. A. Adams, *Sobolev Spaces*, Pure and Applied Mathematics, Vol. 65, Academic Press, New York, London, 1975.

[2] Z. Belhachmi, C. Bernardi, and S. Deparis, *Weighted Clément operator and application to the finite element discretization of the axisymmetric Stokes problem*, Numer. Math., 105 (2006), pp. 217–247.

[3] C. Bernardi, M. Dauge, and Y. Maday, *Spectral Methods for Axisymmetric Domains*, Ser. Appl. Math. (Paris) 3, North–Holland, Amsterdam, 1999.

[4] R. B. Bird, C. F. Curtiss, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Liquids, Volume* 2: *Kinetic Theory*, Wiley Interscience, New York, 1987.

[5] D. Boffi, *Three-dimensional finite element methods for the Stokes problems*, SIAM J. Numer. Anal., 34 (1997), pp. 664–670.

[6] D. Braess, *Finite Elements*, in Theory, Fast Solvers, and Applications in Solid Mechanics, 2nd ed., Cambridge University Press, Cambridge, UK, 2001.

[7] J. H. Bramble and J. E. Pasciak, *Iterative techniques for time dependent Stokes problems*, Comput. Math. Appl., 33 (1997), pp. 13–30.

[8] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, Texts in Appl. Math. 15, 2nd ed., Springer-Verlag, New York, 2002.

[9] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer Ser. Comput. Math. 15, Springer-Verlag, New York, 1991.

[10] D. M. Copeland, J. Gopalakrishnan, and J. Pasciak, *A mixed method for axisymmetric div-curl systems*, Math. Comp., 77 (2008), pp. 1941–1965.

[11] V. Girault and P. A. Raviart, *Finite Element Methods for Navier-Stokes Equations Theory and Algorithms*, Springer Ser. Comput. Math. 5, Springer-Verlag, Berlin, 1986.

[12] A. Jayaraman and A. Belmonte, *Oscillations of a solid sphere falling through a wormlike micellar fluid*, Phys. Rev. E, 67, 2003, pp. 65301–65304.

[13] Y.-J. Lee, J. Wu, J. Xu, and L. Zikatanov, *A sharp convergence estimate of the method of subspace corrections for singular systems*, Math. Comp., 77 (2008), pp. 831–850.

[14] Y.-J. Lee, J. Wu, and J. Chen, *Robust multigrid method for the planar linear elasticity problems*, Numer. Math., 113 (2009), pp. 473–496.

[15] Y. -J. Lee and J. Xu, *New formulations, positivity preserving discretizations and stability analysis for non-Newtonian flow models*, Comput. Methods Appl. Mech. Engrg., 195 (2006), pp. 1180–1206.

[16] Y.-J. Lee and C.-S. Zhang, *Self-sustaining oscillations of the falling sphere through the Johnson-Segalman fluids*, Phys. Rev. Lett., in revision.

[17] Y.-J. Lee, J. Wu, J. Xu, and L, Zikatanov, *Robust subspace correction methods for nearly singular systems*, Math. Models Methods Appl. Sci., 17 (2007), pp. 1937–1963.
[18] H. Li, *Well-posedness, regularity, and finite element analysis for the axisymmetric Laplace operator on polygonal domains*, preprint.
[19] B. Mercier and G. Raguel, *Résolution d'un problème aux limites dans un ouvert axisymmetrique par éléments finis en r, z et séries de Fourier en θ*, RAIRO Anal. Numér, 16 (1982), pp. 405–461.
[20] T. Rusten and R. Winther, *A preconditioned iterative method for saddle point problems*, SIAM J. Matrix Anal. Appl., 13 (1992), pp. 887–904.
[21] L. R. Scott and S. Zhang, *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Math. Comp., 54 (1990), pp. 483–493.
[22] R. Stenberg, *Analysis of mixed finite element methods for the Stokes problems: A unified approach*, Math. Comp., 42 (1984), pp. 9–23.
[23] M. Tabata, *Mixed and stabilized finite element approximations to axisymmetric flow problems*, in Computational Fluid Dynamics, S. Wagner et al., ed., John Wiley & Sons, Chichester, UK, 1994, pp. 176–180.
[24] M. Tabata, *Finite element analysis of axisymmetric flow problems*, Z. Angew. Math. Mech., 76 (1996), pp. 171–174.
[25] J. Xu and L. Zikatanov, *The method of alternating projections and the method of subspace corrections in Hilbert space*, J. Amer. Math. Soc., 15 (2002), pp. 573–597.
[26] L.-A. Ying, *Finite element approximation to axial symmetric Stokes flow*, J. Comput. Math., 4 (1986), pp. 38–49.