Regularity and a priori error analysis of a Ventcel problem in polyhedral domains

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We consider the regularity of a mixed boundary value problem for the Laplace operator on a polyhedral domain, where Ventcel boundary conditions are imposed on one face of the polyhedron and Dirichlet boundary conditions are imposed on the complement of that face in the boundary. We establish improved regularity estimates for the trace of the variational solution on the Ventcel face and use them to derive a decomposition of the solution into a regular and a singular part that belongs to suitable weighted Sobolev spaces. This decomposition, in turn, via interpolation estimates both in the interior as well as on the Ventcel face, allows us to perform an *a priori* error analysis for the finite element approximation of the solution on anisotropic graded meshes. Numerical tests support the theoretical analysis. Copyright © 2016 John Wiley & Sons, Ltd.

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1. Introduction

This article concerns the regularity of solutions to an elliptic boundary value problem for the Laplace operator on a polyhedral domain in $\mathbb{R}^3$ under *Ventcel* or *Wentzell boundary conditions*. The regularity result we establish in weighted Sobolev spaces gives rise, in turn, to *a priori* error estimates for the finite element method (FEM) on a suitable anisotropic mesh.

We first introduce the Ventcel boundary value problem. Let Ω be a bounded domain of $\mathbb{R}^3$ with Lipschitz boundary Γ, and let Γ_V be an open subset of Γ. We denote by $\Gamma_D = \Gamma \setminus \Gamma_V$ the complement of Γ_V, which we assume has also positive measure.

We consider the following mixed boundary value problem:

$$-\Delta u = f, \qquad\qquad \text{in } \Omega, \qquad\qquad (1a)$$

$$u = 0 \qquad\qquad \text{on } \Gamma_D, \qquad\qquad (1b)$$

$$-\Delta_{LB} u + \partial_\nu u = g, \qquad\qquad \text{on } \Gamma_V, \qquad\qquad (1c)$$

where Δ is the standard (Euclidean) Laplacean in $\mathbb{R}^3$, Δ_{LB} is the Laplace–Beltrami operator on Γ, ν is the unit outer normal vector on $\partial\Omega$, ∂_ν means the associated normal derivative, and f and g are given data.

This problem is a special case of a more general boundary value problem, where (1c) is replaced by

$$-\alpha\Delta_{LB} u + \partial_\nu u + \beta u = g,$$

which can be thought of as a generalized Robin-type boundary condition. The more general problem is well posed only under conditions on the sign of α and β. Ventcel boundary conditions arise naturally in many contexts. In the context of multidimensional diffusion processes, Ventcel boundary conditions were introduced in the pioneering work of Ventcel [1,2] (see also the work of Feller for

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one-dimensional processes [3, 4]). They can model heat conduction in materials for which the boundary can store, but not absorb or transmit heat. They can also be derived as approximate boundary conditions in asymptotic problems or artificial boundary conditions in exterior problems (e.g., [5–7] and references therein), in particular in fluid-structure interaction problems.

Problem (1) is known to have a unique variational solution if f and g are in the appropriate Sobolev space as recalled in Section 2. We are concerned here with the higher regularity for solutions to this problem when the data is also regular, in the case that the domain Ω is a polyhedral domain in $\mathbb{R}^3$. It is well known that, because of the presence of edges and corners at the boundary of Ω, even when Γ_V is empty, elliptic regularity does not hold, and the solution is not smooth even if the data is smooth. This loss of regularity affects the rate of convergence of the finite element approximation to the solution if uniform meshes are used.

By using weighted Sobolev spaces, where the weights are the distance to the edges and vertices, respectively, one can characterize precisely the behavior of the variational solution near the singular set in terms of singular function and singular exponents (Theorem 2.5). In turn, the decomposition of the solution into a regular and a singular part, together with interpolation estimates (Theorem 3.1), leads to establishing *a priori* error estimates for the finite element approximation (Corollary 3.4), where the elements are given on an anisotropic mesh that exploits the improved regularity of the solution along the edges versus the corners of the polyhedron. There is a well-established literature on this approach for mixed Dirichlet, Neumann, and even standard Robin boundary condition (e.g., [8–10]). There are also several works in the literature concerning the Ventcel boundary value problems on singular domains (in particular [11, 12]), and their implementation of the FEM (refer to [13] and references therein). The novelty of this work consists in extending the approach using weighted spaces and anisotropic meshes to the Ventcel boundary conditions, which include tangential differential operators at the boundary of the same order as the main operator in Ω. As a matter of fact, the main difficulty in considering such boundary conditions lies in establishing the needed regularity of the traces on the faces of the polyhedron. For simplicity, we restrict here to the case where the Ventcel condition is imposed on only one face of the polyhedron. If the Ventcel condition is imposed on adjacent faces, one would expect higher regularity to hold for the solution on these faces, under suitable transmission conditions at the common edges. However, capturing this behavior entails studying weighted Sobolev spaces for which the weight is the distance to the boundary and not the distance to the singular set (as those arising from the analysis of equations with degenerate coefficients). We reserve to address this problem in future works.

The paper is organized as follows. In Section 2, we recall the variational formulation for Problem (1) and prove our main regularity result for the solution in weighted spaces. In Section 3, we introduce the anisotropic mesh and the associated finite element discretization of the problem, and derive *a priori* error estimates. Section 4 contains some refined 2D interpolation estimates valid on the polyhedral faces, needed for the error analysis. We close in Section 5 by presenting some numerical examples to validate the theoretical analysis.

We end this introduction with some needed notation.

If Ω is a domain of $\mathbb{R}^n$, $n \geq 1$, we employ the standard notation $H^m(\Omega)$ to denote the Sobolev space that consists of functions whose ith derivatives, for $0 \leq i \leq m$, are square integrable. The $L^2(\Omega)$-inner product (norm) will be denoted by $(\cdot, \cdot)_\Omega$ ($\| \cdot \|_\Omega$). The usual norm and semi-norm in $H^s(\Omega)$, for $s \geq 0$, are denoted by $\| \cdot \|_{s,\Omega}$ and $| \cdot |_{s,\Omega}$, respectively. The trace operator from $H^1(\Omega)$ into $H^{\frac{1}{2}}(\partial\Omega)$ will be denoted by γ. We also introduce the space:

$$H^1_{\Gamma_D}(\Omega) = \{u \in H^1(\Omega) : \gamma u = 0 \text{ on } \Gamma_D\},$$

which is clearly a closed subspace of $H^1(\Omega)$. If v is a d-dimensional vector, we will write $v \in H^s(\Omega)^d$, although for ease of notation, we may write $H^s(\Omega)$ simply for $H^s(\Omega)^d$. Lastly, we employ the standard notation $\mathcal{D}'(\Omega)$) to denote the space of distributions on Ω.

Throughout, the notation $A \lesssim B$ is used for the estimate $A \leq C B$, where C is a generic constant that does not depend on A and B. The notation $A \sim B$ means that both $A \lesssim B$ and $B \lesssim A$ hold. We will also employ standard multi-index notation for partial derivatives in $\mathbb{R}^d$, i.e., $\partial^\alpha = \partial^{\alpha_1}_{x_1} \ldots \partial^{\alpha_d}_{x_d}$ where $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d_+$ and $|\alpha| = \alpha_1 + \ldots + \alpha_d$.

2. Some regularity results

In this section, we recall needed facts about the well-posedness of the Ventcel problem (1) and establish regularity estimates for its variational solution in weighted spaces.

The variation formulation of (1) is well known [11, 13, 14]. We let

$$V := \{u \in H^1_{\Gamma_D}(\Omega) : \gamma u \in H^1_0(\Gamma_V)\},$$

which is a Hilbert spaces equipped with the natural norm

$$\|u\|^2_V := \|u\|^2_{1,\Omega} + |\gamma u|_{1,\Gamma_V}, \forall u \in V.$$

We further introduce the bilinear form

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\Gamma_V} \nabla_T(\gamma u) \cdot \nabla_T(\gamma v) \, d\sigma(x), \quad \forall u, v \in V.$$

As this bilinear form is continuous and coercive in V, by the Lax–Milgram lemma, for any $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_V)$, there exists a unique solution $u \in V$ of

$$a(u, v) = \int_\Omega f v \, dx + \int_{\Gamma_V} g v \, d\sigma(x), \quad \forall v \in V. \tag{2}$$

It was shown in [13, Theorem 3.3] that if Γ_D is empty and if Γ is $C^{1,1}$, then u belongs to $H^2(\Omega)$ and γu belongs to $H^2(\Gamma_V)$. This regularity is no longer valid if Ω is a non-convex polyhedral domain, and the main purpose of this section is to describe the behavior of the solution near the singular set, which consists of the edges and vertices of the boundary faces of the polyhedron, and characterize the regularity of boundary traces of u and its derivatives.

To this end, we will employ anisotropic weighted Sobolev spaces, for which the weights are (variants of) the distance to the edges and vertices, respectively. There is a vast literature concerning the use of weighted Sobolev spaces in the analysis of singular domains (we refer for instance to [15–19] and references therein). In the context of the analysis of Dirichlet/Neumann boundary conditions, anisotropic Sobolev spaces were used in [8, 20–22].

From now on, we assume that Ω is a polyhedral domain of the space and that Γ_V is reduced to one face F of the boundary.

By a face, we mean an open face on the boundary. Let $\mathcal{S}$ and $\mathcal{E}$ be the set of vertices and the set of open edges of Ω, respectively.

On the polygonal face F, we denote its set of vertices by $\mathcal{S}_F$. Given a vertex $S \in \mathcal{S}_F$, we denote by (r_S, θ_S) the radial distance and angular component of the local polar coordinate system centered at S on the plane containing F. In addition, we let $\omega_{F,S}$ be the interior angle on the face F associated with the vertex S.

Following [8], we consider a triangulation $\{\Lambda_\ell\}_{\ell=1}^L$ of the domain Ω that consists of disjoint tetrahedra Λ_ℓ. We will refer to each tetrahedron Λ_ℓ as a *macro element*, to distinguish it from the elements of the mesh utilized in the analysis of the FEM in Section 3. The purpose of the macro elements is to localize the construction and the regularity estimates near edges and vertices of Ω. We will also refer to any edge or vertex of an element Λ_ℓ as a *singular edge* or *singular vertex*, if that edge or vertex lies along a true edge or is a true vertex of Ω and the solution is not in H^2 near that true edge or vertex.

We will assume that each Λ_ℓ contains at most one singular edge and at most one singular vertex. If Λ_ℓ contains both a singular edge and a singular vertex, that vertex belongs to that edge. We will also assume that all Λ_ℓ are shape regular with diameter of order $O(1)$. In each macro element Λ_ℓ, we introduce a local Cartesian coordinate system $x^{(\ell)} = \left(x_1^{(\ell)}, x_2^{(\ell)}, x_3^{(\ell)}\right)$, such that the singular vertex, if it exists, is at the origin, and the singular edge, if it exists, lies along the $x_3^{(\ell)}$-axis. We then define the distance functions to the set of singular edges and singular vertices, (3a) respectively, as follows:

$$r^{(\ell)}(x^{(\ell)}) = \sqrt{\left(x_1^{(\ell)}\right)^2 + \left(x_2^{(\ell)}\right)^2}, \tag{3a}$$

$$R^{(\ell)}(x^{(\ell)}) = \sqrt{\left(x_1^{(\ell)}\right)^2 + \left(x_2^{(\ell)}\right)^2 + \left(x_3^{(\ell)}\right)^2}, \tag{3b}$$

and introduce the auxiliary function

$$\theta^{(\ell)}\left(x^{(\ell)}\right) = r^{(\ell)}\left(x^{(\ell)}\right) / R^{(\ell)}\left(x^{(\ell)}\right). \tag{3c}$$

We observe that $r^{(\ell)}$ and $R^{(\ell)}$ extend as continuous functions to the closure of the macro element Λ_ℓ, while $\theta^{(\ell)}$ extends as a bounded function.

In what follows, we will omit the sup-index (ℓ) in these distance functions and in x, when there is no confusion about the underlying macro element. Given a subdomain $\Lambda \subset \Omega$, we define the following weighted Sobolev space for $k \in \mathbb{N}$ and $\beta, \delta \in \mathbb{R}$:

$$V_{\beta,\delta}^k(\Lambda) := \left\{ v \in \mathcal{D}'(\Lambda), \, \|v\|_{V_{\beta,\delta}^k(\Lambda)} < \infty \right\},$$

where

$$\|v\|_{V_{\beta,\delta}^k(\Lambda)}^2 = \sum_{|\alpha| \leq k} \left\| R^{\beta-k+|\alpha|} \theta^{\delta-k+|\alpha|} \partial^{|\alpha|} v \right\|_{L^2(\Lambda)}^2,$$

and $R(x) = R^{(\ell)}(x^{(\ell)})$ and $\theta(x) = \theta^{(\ell)}(x^{(\ell)})$, given in (3b) and in (3c), if $x \in \Lambda_\ell$ is represented by $x^{(\ell)}$ in local coordinates.

We will also need to define spaces on the faces of Ω. To this end, given G a bounded polygonal domain in $\mathbb{R}^2$, we also define

$$V_\gamma^k(G) := \{v \in \mathcal{D}'(G), \, \|v\|_{V_\gamma^k(G)} < \infty\},$$

where

$$\|v\|_{V_\gamma^k(G)}^2 = \sum_{|\alpha| \leq k} \left\| \rho^{\gamma-k+|\alpha|} \partial^{|\alpha|} v \right\|_{L^2(G)}^2.$$

In the preceding equations, ρ is the distance function to the set of vertices of G, defined in a manner similar to R.

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We further classify the initial macro elements Λ_ℓ into four types as follows:

Type 1. $\bar{\Lambda}_\ell \cap (\mathcal{S} \cup \mathcal{E}) = \emptyset$;
Type 2. $\bar{\Lambda}_\ell \cap \mathcal{S} \neq \emptyset$ but $\bar{\Lambda}_\ell \cap \mathcal{E} = \emptyset$;
Type 3. $\bar{\Lambda}_\ell \cap \mathcal{E} \neq \emptyset$ but $\bar{\Lambda}_\ell \cap \mathcal{S} = \emptyset$;
Type 4. $\bar{\Lambda}_\ell \cap \mathcal{E} \neq \emptyset$ and $\bar{\Lambda}_\ell \cap \mathcal{S} \neq \emptyset$.

We first start with an improved regularity of $\partial_\nu u$ on Γ_V. In what follows, for ease of notation, we will let u_F be the trace γu of u on the face F. Furthermore, for a two-dimensional domain D, we define the space $\tilde{H}^s(D)$, $0 < s < 1$, as the closure of $C_c^\infty(D)$ in $H^s(D)$.

Lemma 2.1
If $D \subset \mathbb{R}^2$ is a two-dimensional domain with Lipschitz boundary, then for any $h \in (\tilde{H}^{\frac{1}{2}}(D))'$, the unique solution $w \in H_0^1(D)$ of

$$-\Delta w = h \text{ in } D,$$

belongs to $H^{1+\varepsilon}(D) \cap H_0^1(D)$ for any $\varepsilon \in \left(0, \frac{1}{2}\right)$.

Proof
We fix $\varepsilon \in \left(0, \frac{1}{2}\right)$. Because $H_0^{1-\varepsilon}(D) = \tilde{H}^{1-\varepsilon}(D)$ is continuously and densely embedded into $\tilde{H}^{\frac{1}{2}}(D)$, by duality, we obtain that $\left(\tilde{H}^{\frac{1}{2}}(D)\right)'$ is continuously embedded into $\left(H_0^{1-\varepsilon}(D)\right)' = H^{-1+\varepsilon}(D)$. Hence, w can be seen as a solution of the Laplace equation with datum in $H^{-1+\varepsilon}(D)$. Owing to Theorem 18.13 and Remark 18.17/2 in [15], w belongs to $H^{1+\varepsilon}(D)$. $\qquad\square$

Lemma 2.2
Let $u \in V$ be the solution of (2), then we have

$$\partial_\nu u \in L^2(\Gamma_V).$$

Proof
We first observe that, by Theorem 2.8 of [23], $\partial_\nu u \in (\tilde{H}^{\frac{1}{2}}(\Gamma_V))'$. Then, we may interpret $u_F \in H_0^1(F)$ as the unique variational solution of

$$\Delta_{LB} u_F = -g + \partial_\nu u \in \left(\tilde{H}^{\frac{1}{2}}(F)\right)'. \tag{4}$$

By Lemma 2.1, we deduce that u_F belongs to $H^{1+\varepsilon}(F) \cap H_0^1(F)$ for any $\varepsilon \in (0, \frac{1}{2})$. We now fix $\varepsilon \in \left(0, \frac{1}{2}\right)$ small enough that the mapping

$$H^{\frac{3}{2}+\varepsilon}(\Omega) \cap H_0^1(\Omega) \to H^{-\frac{1}{2}+\varepsilon}(\Omega) : v \to \Delta v,$$

is an isomorphism (we refer to [15, Theorem 18.13]).

Now, by applying the trace theorem from [24], there exists $w \in H^{\frac{3}{2}+\varepsilon}(\Omega)$ such that

$$\gamma w = 0 \text{ on } \Gamma_D, \tag{5}$$

$$\gamma w = u_F \text{ on } F. \tag{6}$$

This implies, again by uniqueness, that $v := u - w \in H_0^1(\Omega)$ is the solution of

$$-\Delta v = f + \Delta w \in H^{-\frac{1}{2}+\varepsilon}(\Omega).$$

We therefore deduce that v belongs to $H^{\frac{3}{2}+\varepsilon}(\Omega)$ and, hence, u belongs to this space as well. By a standard trace theorem, we finally conclude that $\nabla u \in H^\varepsilon(\Gamma)^3$. $\qquad\square$

Thus, we have the following decomposition of the singular solution u_F on the polygonal face.

Corollary 2.3
Let again $u \in V$ be the solution of (2). Then, it holds

$$u_F = u_{F,R} + \sum_{S \in \mathcal{S}_F : \omega_{F,S} > \pi} c_S r_S^{\frac{\pi}{\omega_{F,S}}} \sin\left(\frac{\pi \theta_S}{\omega_{F,S}}\right), \tag{7}$$

where $u_{F,R} \in H^2(F)$ and $c_S \in \mathbb{C}$.

Proof
As $\partial_\nu u$ belongs to $L^2(F)$ by Lemma 2.2, the right-hand side in (4) is now in $L^2(F)$, and therefore Theorem 4.4.3.7 of [25] yields (7). $\qquad\square$

We will refer to $u_{F,R}$ as the *regular part* of u_F, hence the subscript, as it has the expected regularity from elliptic theory. We will consequently call $u_F - u_{F,R}$ the *singular part* of u_F.

For the regularity of the solution in the interior of the domain Ω, we first have the following lifting estimate based on the trace theorem.

Lemma 2.4
Given $u_F \in H^{\frac{3}{2}+\varepsilon}(F) \cap H^1_0(F)$ for some $\varepsilon \in (0, \frac{1}{2})$, there exists a lifting $w \in H^{2+\varepsilon}(\Omega)$ satisfying (5)–(6).

Proof
The idea is to use again the trace theorem from [24] with $s = 2 + \varepsilon$ and the operator

$$u \to (f_{j,0}, f_{j,1})^N_{j=1} := \left(u_{|\Gamma_j}, (\partial_{\nu_j} u)_{|\Gamma_j} \right)^N_{j=1}, \tag{8}$$

where Γ_j are the faces of Ω and ν_j the outward normal vector along Γ_j. As $1 + \varepsilon$ is not an integer, this trace operator (8) is surjective from $H^{2+\epsilon}(\Omega)$ onto the subspace of $\prod^N_{j=1} \left(H^{\frac{3}{2}+\varepsilon}(\Gamma_j) \times H^{\frac{1}{2}+\varepsilon}(\Gamma_j) \right)$ that satisfies the compatibility conditions (C_1) of [24]. If we assume that $\Gamma_1 = F$, it is therefore sufficient to show that there exist $f_{j,1} \in H^{\frac{1}{2}+\varepsilon}(\Gamma_j), j = 1, \cdots, N$ such that

$$(u_F, f_{1,1}) \times (0, f_{j,1})^N_{j=2}$$

satisfies these conditions (C_1). Because such conditions are quite technical to check, as in [24], we can reduce to check such conditions in the case where Ω is the trihedral $x_i > 0$, $i = 1, 2, 3$ and F is the face $x_1 = 0$ (and hence $N = 3$ with $\Gamma_2 \equiv x_2 = 0$ and $\Gamma_3 \equiv x_3 = 0$), by means of a localization argument and a linear change of variables. In such a case, the conditions (C_1) of [24] for $(u_F, f_{1,1}) \times (0, f_{j,1})^3_{j=2}$ take the form:

$$u_F = 0 \text{ on } A_{1,2} \cup A_{2,3}, \tag{9a}$$
$$f_{1,1} = 0 \text{ on } A_{1,3}, \tag{9b}$$
$$\partial_2 u_F = 0 \text{ on } A_{1,3}, \tag{9c}$$
$$\partial_3 u_F = f_{3,1} \text{ on } A_{1,3}, \tag{9d}$$
$$f_{1,1} = 0 \text{ on } A_{1,2}, \tag{9e}$$
$$\partial_2 u_F = f_{2,1} \text{ on } A_{1,2}, \tag{9f}$$
$$\partial_3 u_F = 0 \text{ on } A_{1,2}, \tag{9g}$$
$$f_{2,1} = 0 \text{ on } A_{2,3}, \tag{9h}$$
$$f_{3,1} = 0 \text{ on } A_{2,3}, \tag{9i}$$

where $A_{i,j} = \bar{\Gamma}_i \cap \bar{\Gamma}_j$. The first condition trivially holds as u_F belongs to $H^1_0(F)$, and similarly (9c) (respectively (9g)) because $\partial_2 u_F$ ($\partial_3 u_F$) is the tangential derivatives of u_F on $A_{1,3}$ ($A_{1,2}$). To satisfy the second and fourth conditions, we simply take $f_{1,1} = 0$. Hence, it remains to verify the conditions (9f) and (9h) (respectively (9d) and (9i)) that can be interpreted as constraints on $f_{2,1}$ and $f_{3,1}$, respectively. In other words, we look for $f_{2,1} \in H^{\frac{1}{2}+\varepsilon}(\Gamma_2)$ (respectively $f_{3,1} \in H^{\frac{1}{2}+\varepsilon}(\Gamma_3)$) satisfying the boundary conditions (9f) and (9h) (respectively (9d) and (9i)). Such a solution $f_{2,1}$ (and similarly $f_{3,1}$) exists by applying Theorem 1.5.1.2 of [25] (valid for a quarter plane), because the function defined by $\partial_2 u_F$ on $A_{1,2}$ and 0 on $A_{2,3}$ belongs to $H^\varepsilon(\Gamma_2)$. $\square$

For a vertex $v \in \mathcal{S}$, let C_v be the infinite polyhedral cone that coincides with Ω in the neighborhood of v. Let $G_v = C_v \cap S^2(v)$ be the intersection of C_v and the unit sphere centered at v. For an edge $e \in \mathcal{E}$, let ω_e be the interior angle between the two faces of Ω that contain e. Then, for $v \in \mathcal{S}$ and for $e \in \mathcal{E}$, respectively, we define the following parameters associated to the singularities in the solution near v and e:

$$\lambda_v := -\frac{1}{2} + \sqrt{\lambda_{v,1} + \frac{1}{4}}, \qquad \lambda_e := \pi/\omega_e, \tag{10}$$

where $\lambda_{v,1}$ is the smallest positive eigenvalue of the Laplace–Beltrami operator on G_s with Dirichlet boundary conditions. We observe that a vertex v is singular if $\lambda_v < 1/2$ and an edge e is singular if $\lambda_e < 1$. For a given macro element Λ_ℓ, we set $\lambda_v^{(\ell)} = \lambda_v$ if Λ_ℓ contains one singular vertex v of Ω and $\lambda_v^{(\ell)} = \infty$ otherwise. Similarly, we set $\lambda_e^{(\ell)} = \lambda_e$ if Λ_ℓ contains on singular edge e of Ω and $\lambda_e^{(\ell)} = \infty$ otherwise. Then, the following decomposition for the variational solution of (1) holds.

Theorem 2.5
Let $u \in V$ be again the solution of (2). We have

$$u = u_R + u_S, \tag{11}$$

where $u_R \in H^2(\Omega)$ and $u_S \in H^1(\Omega)$ satisfies, for all $\ell \in \{1, \ldots, L\}$,

$$\frac{\partial u_S}{\partial x_j^{(\ell)}} \in V^1_{\beta,\delta}(\Lambda_\ell), j = 1, 2, \tag{12}$$

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$$\frac{\partial u_S}{\partial x_3^{(\ell)}} \in V_{\beta,0}^1(\Lambda_\ell), \tag{13}$$

for any $\beta, \delta \geq 0$ such that

$$\beta > \frac{1}{2} - \lambda_v^{(\ell)}, \qquad \delta > 1 - \lambda_e^{(\ell)},$$

Again, the subscripts refer to the fact that u_R has the expected regularity, and hence it will be called the *regular part* of the solution, while $u_S = u - u_R$ represents the *singular part*.

Proof

The decomposition (7) implies that there exists $\varepsilon \in (0, \frac{1}{2})$ small enough such that $u_F \in H^{\frac{3}{2}+\varepsilon}(F) \cap H_0^1(F)$. Hence, by Lemma 2.4, there exists a lifting $w \in H^{2+\varepsilon}(\Omega)$ satisfying (5)–(6). With this lifting at hands, we consider $u - w$, which belongs to $H_0^1(\Omega)$ and is the weak solution of

$$-\Delta(u - w) = f + \Delta w.$$

As $f + \Delta w$ belongs to $L^2(\Omega)$, we can apply Theorem 2.10 of [8] to $u - w$, which gives the decomposition:

$$u - w = u_R + u_S,$$

with $u_R \in H^2(\Omega)$ and u_S satisfying (12)–(13). Finally, the result follows by setting $u_R = w + u_r$. $\square$

Remark 2.6

Theorem 2.5 shows that for the solution to (2) with the Ventcel boundary condition, its regularity in Ω, determined by the geometry of the domain, is similar to the regularity of the Poisson equation with the Dirichlet boundary condition. Meanwhile, the trace of the solution u on the face F is the solution of a two-dimensional elliptic problem with the Dirichlet boundary condition. Corollary 2.3 implies that the regularity of the trace depends on the interior angles of the polygon F.

3. Finite element approximation

We consider an (anisotropic) triangulation $\mathcal{T}_h = \{T_i\}_{i=1}^N$ of Ω as in Section 3 of [8] or in Section 2 of [21], consisting of tetrahedra with refinement parameters μ_ℓ and ν_ℓ. We assume the general conditions for a triangulation of the domain (e.g., [26, 27]) and that the number of tetrahedra m satisfies $N \sim h^{-3}$, where h is the global mesh size. In addition, we assume that the initial subdomains Λ_ℓ are resolved exactly, namely, $\bar{\Lambda}_\ell = \cup_{i \in L_\ell} \bar{T}_i$, where $\ell = 1, \cdots, L$ and $L_\ell \subset \{1, \cdots, m\}$ is the index set of the tetrahedra included in $\bar{\Lambda}_\ell$.

In each Λ_ℓ, the parameters $\mu_\ell, \nu_\ell \in (0, 1]$ determine the anisotropic mesh refinement close to edges and vertices, respectively as indicated in (14). When $\mu_\ell = 1$ or $\nu_\ell = 1$, there will be no graded refinement in Λ_ℓ. We recall the local Cartesian coordinate system $\left(x_1^{(\ell)}, x_2^{(\ell)}, x_3^{(\ell)}\right)$ in each of the subdomain Λ_ℓ, which is such that the singular vertex is at the origin and the singular edge is along the x_3-axis, if they exist. Then, for each element $T_i \subset \Lambda_\ell$ of the triangularization, we let

$$r_i := \inf_{x \in T_i} \left[\left(x_1^{(\ell)}\right)^2 + \left(x_2^{(\ell)}\right)^2\right]^{1/2}, \qquad R_i := \inf_{x \in T_i} \left[\left(x_1^{(\ell)}\right)^2 + \left(x_2^{(\ell)}\right)^2 + \left(x_3^{(\ell)}\right)^2\right]^{1/2},$$

be the distance of T_i to the x_3-axis and the origin, respectively. We then introduce local, anisotropic mesh parameters in T_i as follows:

$$h_i := \begin{cases} h^{1/\mu_\ell} & \text{if } r_i = 0, \\ hr_i^{1-\mu_\ell} & \text{if } r_i > 0, \end{cases} \qquad H_i := \begin{cases} h^{1/\nu_\ell} & \text{if } 0 \leq R_i \lesssim h^{1/\nu_\ell}, \\ hR_i^{1-\nu_\ell} & \text{if } R_i \gtrsim h^{1/\nu_\ell}, \end{cases} \tag{14}$$

We also introduce the actual mesh sizes $\tilde{h}_{j,i}$, which are the lengths of the projections of $T_i \subset \Lambda_\ell$ on the $x_j^{(\ell)}$-axis, $1 \leq j \leq 3$. Then, there exists a triangulation $\mathcal{T}_h$ satisfying the following conditions:

1. If $\mu_\ell < 1$, then $\tilde{h}_{j,i} \sim h_i, j = 1, 2, \tilde{h}_{3,i} \lesssim H_i$, and $\tilde{h}_{3,i} \sim H_i$ if $r_i = 0$.
2. The number of tetrahedra in Λ_ℓ with $r_i = 0$ is of order h^{-1}.
3. The number of tetrahedra in Λ_ℓ such that $0 \leq R_i \lesssim h^{1/\nu_\ell}$ is bounded by $h^{2\mu_\ell/\nu_\ell - 2}$, and there is only one tetrahedral element with $R_i = 0$.
4. If $\mu_\ell < 1$, then $\mu_\ell \leq \nu_\ell$ for $1 \leq \ell \leq L$.

We refer to [8] for a detailed description of these conditions. It is clear that this triangulation $\mathcal{T}_h$ induces an exact triangulation $\mathcal{F}_h$ of the face F, the elements of which are simply given by $\bar{T} \cap \bar{F}$ for $T \in \mathcal{T}_h$.

Based on these triangulations, we introduce the approximation space V_h of V as follows:

$$V_h := \left\{u_h \in V : u_{|T} \in \mathbb{P}_1(T), \forall T \in \mathcal{T}_h\right\},$$

where $\mathbb{P}_m, m \in \mathbb{Z}_+$, denotes the space of all polynomials of degree $\leq m$. This is clearly a closed subspace of V.

Then, the finite element approximation of Problem (2) consists of looking for a solution $u_h \in V_h$ of

$$a(u_h, v_h) = \int_\Omega f v_h \, dx + \int_{\Gamma_V} g v_h \, d\sigma(x), \qquad \forall v_h \in V_h. \tag{15}$$

By Céa's lemma, we have

$$\|u - u_h\|_V \lesssim \inf_{v_h \in V_h} \|u - v_h\|_V,$$

Hence, an error estimate will be available if we can built an appropriate approximation v_h of u. This is the purpose of the next theorems in this section.

Theorem 3.1
Recall the parameters in (10) and in (14). Assume that for all $\ell = 1, \cdots, L$, we have

$$\mu_\ell < \lambda_e^{(\ell)}, \tag{16a}$$

$$\nu_\ell < \lambda_v^{(\ell)} + \frac{1}{2} \tag{16b}$$

$$\frac{1}{\nu_\ell} + \frac{1}{\mu_\ell}\left(\lambda_v^{(\ell)} - \frac{1}{2}\right) > 1. \tag{16c}$$

Then, there exists $v_h \in V_h$ such that

$$\|u - v_h\|_{1,\Omega} \lesssim h. \tag{17}$$

Proof
The proof of Theorem 2.5 furnishes the splitting of u as

$$u = \tilde{u} + w,$$

where $\tilde{u} = u - w$ and $w \in H^{2+\varepsilon}(\Omega)$ with $\varepsilon \in \left(0, \frac{1}{2}\right)$. Hence we define an interpolant $I_h u$ of u as follows:

$$I_h u := \tilde{u}_I + D_h(\tilde{u} - \tilde{u}_I) + L_h w, \tag{18}$$

where D_h is the interpolant introduced in [21], $\tilde{u}_I$ is the Lagrange interpolant of $\tilde{u}$ with respect to the partition $\{\Lambda_\ell\}$, while $L_h w$ is the standard Lagrange interpolant of w, which consists of piece-wise polynomials of degree 1. Then, using the regularity estimate in Theorem 2.5 and applying Theorem 3.11 of [21], we have

$$\|\tilde{u} - \tilde{u}_I + D_h(\tilde{u} - \tilde{u}_I)\|_{1,\Omega} \lesssim h. \tag{19}$$

On the other hand, as w belongs to $H^{2+\varepsilon}(\Omega)$ and $H^{2+\varepsilon}(\Omega)$ is continuously embedded into $W^{2,p}(\Omega)$ with $p \in \left(2, \frac{6}{3-2-\varepsilon}\right)$, by the estimate (5.6) of [8] for a fixed $p \in \left(2, \frac{6}{3-2-\varepsilon}\right)$, we deduce that

$$\|w - L_h w\|_{1,\Omega} \lesssim h\|w\|_{W^{2,p}(\Omega)} \lesssim h\|w\|_{2+\varepsilon,\Omega}.$$

This estimate and (19) prove the estimate (17). □

We observe that $I_h u = L_h w = L_h u_F$ on the face F, because $\tilde{u}_I$ and $D_h(\tilde{u} - \tilde{u}_I)$ vanish on F. We next state and prove an error estimate for the finite element approximation on the face F.

Theorem 3.2
For a macro element Λ_ℓ such that $\bar{\Lambda}_\ell \cap F \neq \emptyset$, let $\omega_{F,v,\ell}$ be the interior angle of F associated with the vertex $v \in \mathcal{V}$. Assume that the conditions

$$\nu_\ell < \frac{\pi}{\omega_{F,v,\ell}}, \tag{20a}$$

$$\frac{1}{\nu_\ell} + \frac{1}{\mu_\ell}\left(\frac{\pi}{\omega_{F,v,\ell}} - 1\right) > 1, \tag{20b}$$

are satisfied. Then, it holds:

$$\|u_F - L_h u_F\|_{1,F} \lesssim h. \tag{21}$$

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Proof

We will prove that for all $\ell = 1, \cdots, L$, we have

$$|u_F - L_h u_F|_{1, F \cap \bar{\Lambda}_\ell} \lesssim h. \tag{22}$$

Hence, summing on ℓ, we find that

$$|u_F - L_h u_F|_{1, F} \lesssim h,$$

and the conclusion of the theorem follows from Poincaré's inequality.

To prove (22), we distinguish different cases:

1. $\bar{F} \cap \bar{\Lambda}_\ell$ contains no singular vertex or singular edge: In this case, u_F belongs to $H^2(F \cap \bar{\Lambda}_\ell)$ and the mesh on $F \cap \bar{\Lambda}_\ell$ is quasi-uniform. Thus, the estimate (22) is standard.

2. $\bar{F} \cap \bar{\Lambda}_\ell$ contains a singular vertex v but no singular edge: Thus, u_F belongs to

$$V_\gamma^2(F \cap \bar{\Lambda}_\ell) = \{v \in L^2_{\mathrm{loc}}(F \cap \bar{\Lambda}_\ell) : R^{\gamma + |\beta| - 2} D^\beta v \in L^2(F \cap \bar{\Lambda}_\ell), \forall |\beta| \leq 2\},$$

with $\gamma > 1 - \frac{\pi}{\omega_{F, v, \ell}}$, and the estimate (22) is also standard, because the triangulation in $F \cap \bar{\Lambda}_\ell$ is isotropic (see for instance [28], [25, §8.4]).

3. $\bar{F} \cap \bar{\Lambda}_\ell$ contains a singular edge: Then, the mesh on $F \cap \bar{\Lambda}_\ell$ is anisotropic. There are two possible situations: (S1) $\bar{F} \cap \bar{\Lambda}_\ell$ contains no singular vertex; and (S2) $\bar{F} \cap \bar{\Lambda}_\ell$ also contains a singular vertex v. Because of Corollary 2.3, for (S1), u_F belongs to $H^2(F \cap \bar{\Lambda}_\ell)$, while for (S2), u_F belongs to $V_\gamma^2(F \cap \bar{\Lambda}_\ell)$. Now for any triangle T_i in $F \cap \bar{\Lambda}_\ell$, we will prove that

$$|u_F - L_h u_F|_{1, T_i} \lesssim h |u_F|_{2, \gamma, T_i}, \tag{23}$$

with $\gamma = 0$ for (S1) and $\gamma > 1 - \frac{\pi}{\omega_{F, v, \ell}}$ for (S2), where

$$|u|^2_{2, \gamma, T} = \sum_{|\alpha| = 2} \int_T R^{2\gamma} |D^\alpha u|^2 \, dx.$$

If this estimate is valid, then summing on T_i, we get (22).

To prove (23), we distinguish two cases.

(i) If T_i is far from the singular corner, then we know that u_F belongs to $H^2(T_i)$, and, by using Estimate (25), we have

$$\begin{aligned} |u_F - L_h u_F|_{1, T_i} &\lesssim h_i |\partial_1 u_F|_{1, T_i} + H_i |\partial_3 u_F|_{1, T_i} \\ &\lesssim H_i |u_F|_{2, T_i}. \end{aligned} \tag{24}$$

If Λ_ℓ is of Type 3, then u_F belongs to $H^2(F \cap \bar{\Lambda}_\ell)$, but as $\nu_\ell = 1$, by the assumptions on the mesh, we have $H_i \lesssim h$; hence, the estimate (24) directly yields (23). If Λ_ℓ is of Type 4, we again distinguish two cases:

(a) If $R_i \gtrsim h^{\frac{1}{\nu_\ell}}$, then $H_i \sim h R_i^{1 - \nu_\ell}$, and therefore,

$$|u_F - L_h u_F|_{1, T_i} \lesssim h R_i^{1 - \nu_\ell} |u_F|_{2, T_i}.$$

This yields (23) by our assumption (20a).

(b) If $0 < R_i \lesssim h^{\frac{1}{\nu}}$, then as $H_i \sim h^{\frac{1}{\nu}}$, the estimate (24) becomes

$$|u_F - L_h u_F|_{1, T_i} \lesssim h^{\frac{1}{\nu}} |u_F|_{2, T_i}.$$

But from Lemma 4.5, we know that $R_i \gtrsim h^{\frac{1}{\mu}}$ and, therefore,

$$|u_F - L_h u_F|_{1, T_i} \lesssim h^{\frac{1}{\nu} - \frac{\gamma}{\mu}} R_i^\gamma |u_F|_{2, T_i}.$$

This yields (23) by our assumption (20b).

(ii) If T_i is near a singular corner (i.e., $R_i = 0$), then applying Lemma 4.3 we have

$$|u_F - L_h u_F|_{1, T_i} \lesssim h^{\frac{1}{\nu} - \frac{\gamma}{\mu}} |u_F|_{2, \gamma, T_i}.$$

Again we get (23) owing to our assumption (20b).

The proof is now complete. $\qquad\qquad\square$

Theorems 3.1 and 3.2 directly lead to the following *a priori* global interpolation estimate on u and error estimate on the Finite Element solution u_h.

Corollary 3.3
Assume that for all $\ell = 1, \cdots, L$, (16a), (16b), (16c), (20a), and (20b) hold. Then there exists $v_h \in V_h$ such that

$$\|u - v_h\|_V \lesssim h.$$

Corollary 3.4
Under the assumption of Corollary 3.3, if $u \in V$ is the solution of (2) and $u_h \in V_h$ the solution of (15), then

$$\|u - u_h\|_V \lesssim h.$$

4. Anisotropic error estimates in two dimension

To complete the proof of Theorem 3.2, we need some interpolation estimates in two space dimensions. In this section, T_i will be a triangle in the triangulation $\mathcal{F}_h$ of the face F, which is induced by the triangulation $\mathcal{T}_h$ of Ω. We will need the two-dimensional version of Theorem 4.10 of [8], given as follows.

Theorem 4.1
Assume that Λ_ℓ is of Type 3 or 4. Suppose that $\bar{F} \cap \bar{\Lambda}$ contains the singular edge. Recall the local Cartesian coordinate system (x_1, x_2, x_3) for Λ_ℓ, for which the singular edge is on the x_3-axis. Let $F \cap \bar{\Lambda}$ be in the plane given by $x_2 = 0$. Let $T_i \subset F \cap \bar{\Lambda}_\ell$ be a triangle in the triangulation $\mathcal{F}_h$. Then, for $v \in H^2(T_i)$, we have

$$|v - L_h v|_{1,T_i} \lesssim h_i |\partial_1 v|_{1,T_i} + H_i |\partial_3 v|_{1,T_i}, \tag{25}$$

where h_i and H_i are defined in (14).

Proof
Let $\tilde{h}_{1,i}$ and $\tilde{h}_{3,i}$ be the lengths of the projections of T_i on the x_1-axis and x_3-axis, respectively. We distinguish between the case $\tilde{h}_{3,i} \lesssim h_i$ or not.

1. If $\tilde{h}_{3,i} \lesssim h_i$, then diam $T_i \lesssim h_i$ (see the proof of Theorem 4.10 of [8]) and owing to Theorem 2 of [20], we have

$$|v - L_h v|_{1,T_i} \lesssim h_i |v|_{2,T_i},$$

   and (25) holds since $h_i \lesssim H_i$.
2. If $\tilde{h}_{3,i} \gtrsim h_i$, then Theorem 1 of [20] on the reference element and Lemma 4.8 of [8] yield

$$|v - L_h v|_{1,T_i} \lesssim \tilde{h}_{i,1} |\partial_1 v|_{1,T_i} + \tilde{h}_{3,i} |\partial_3 v|_{1,T_i}.$$

   This estimate implies (25), because $\tilde{h}_{i,1} \lesssim h_i$ and $\tilde{h}_{3,i} \lesssim H_i$ (see assumption (B1) in [20], recalling that $\mu_\ell < 1$ if a macro element is of Type 3 or 4). □

We continue with an anisotropic error estimate in weighted Sobolev spaces (compare with Theorem 1 of [20] for two-dimensional triangles in standard Sobolev spaces and Theorem 4.5 of [8] for three-dimensional tetrahedra in weighted Sobolev spaces).

Theorem 4.2
Let $\hat{T}$ be the standard reference element of vertices $(0,0)$, $(1,0)$, and $(0,1)$. Denote by $\hat{R}$ the distance to $(0,0)$. Let $0 \le \gamma < 1$. Then for all $u \in V_\gamma^2(\hat{T})$, and $i = 1$ or 2, we have

$$\|\partial_i(u - Lu)\|_{0,\hat{T}} \lesssim \|\hat{R}^\gamma \nabla \partial_i u\|_{0,\hat{T}}, \tag{26}$$

where Lu is the Lagrange interpolant of u.

Proof
We first remark that Lemma 8.4.1.2 of [25] shows that $V_\gamma^2(\hat{T})$ is continuously embedded into $C(\bar{\hat{T}})$, hence the Lagrange interpolant Lu of u is well-defined. We define the space:

$$H^{1,\gamma}(\hat{T}) := \left\{ v \in L^2(\hat{T}) : \hat{R}^\gamma \nabla v \in L^2(\hat{T})^2 \right\},$$

which is an Hilbert space equipped with its natural norm $\|\cdot\|_{1,\gamma}$. We will also use the semi-norm:

$$|v|_{1,\gamma} = \|\hat{R}^\gamma \nabla v\|_{\hat{T}}, \quad \forall v \in H^{1,\gamma}(\hat{T}).$$

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Then by the proof of Lemma 8.4.1.2 of [25], we know that $H^{1,\gamma}(\hat{T})$ is embedded into $W^{1,p}(\hat{T})$ for all $1 < p < \frac{2}{1+\gamma}$, and hence compactly embedded into $L^2(\hat{T})$. The first embedding and a trace theorem also guarantee that any $v \in H^{1,\gamma}(\hat{T})$ satisfies

$$v \in L^1(\hat{e}), \qquad \|v\|_{L^1(\hat{e})} \lesssim \|v\|_{1,\gamma}, \tag{27}$$

for any edge $\hat{e}$ of $\hat{T}$. The second embedding implies that

$$\|v\|_{1,\gamma} \lesssim |v|_{1,\gamma}, \tag{28}$$

for all $v \in H^{1,\gamma}(\hat{T})$ such that $\int_{\hat{T}} v \, dx = 0$.

Now, we follow the arguments of Lemma 3 and Theorem 1 of [20]. We will first prove the estimate for ∂_1. We observe that (27) implies that the functional

$$F(v) = \int_{\hat{e}_1} v(x) \, d\sigma,$$

where $\hat{e}_1$ is the edge of $\hat{T}$ parallel to the $\hat{x}_1$ axis, is well-defined, and continuous on $H^{1,\gamma}(\hat{T})$:

$$|F(v)| \lesssim \|v\|_{1,\gamma}, \forall v \in H^{1,\gamma}(\hat{T}). \tag{29}$$

Next, we note that

$$F(\partial_1(u - Lu)) = 0,$$

if $u \in V_\gamma^2(\hat{T})$. We then define the polynomial q of degree 1 by

$$q(\hat{x}_1, \hat{x}_2) = c\hat{x}_1,$$

where

$$c = 2 \int_{\hat{T}} (\partial_1 u)(\hat{x}) \, d\hat{x}.$$

With this choice, we see that

$$\int_{\hat{T}} (\partial_1(u - q))(\hat{x}) \, d\hat{x} = 0,$$

and therefore by (28) we obtain

$$\|\partial_1(u - q)\|_{1,\gamma} \lesssim |\partial_1(u - q)|_{1,\gamma} = |\partial_1 u|_{1,\gamma}. \tag{30}$$

As $q - Lu$ is linear, $\partial_1(q - Lu)$ is constant, and we can write

$$\|\partial_1(q - Lu)\|_{1,\gamma} \lesssim |F(\partial_1(q - Lu))| = |F(\partial_1(q - u))|.$$

By (29), we deduce that

$$\|\partial_1(q - Lu)\|_{1,\gamma} \lesssim \|\partial_1(q - u)\|_{1,\gamma}.$$

This estimate and the triangle inequality imply that

$$\|\partial_1(u - Lu)\|_{0,\hat{T}} \leq \|\partial_1(u - Lu)\|_{1,\gamma} \leq \|\partial_1(u - q)\|_{1,\gamma} + \|\partial_1(q - Lu)\|_{1,\gamma} \lesssim \|\partial_1(q - u)\|_{1,\gamma}$$

and the conclusion for ∂_1 follows from (30). The estimate for ∂_2 follows in an analogous manner. $\square$

Then, we are ready to derive the interpolation error estimate near a singular corner of F.

Lemma 4.3
Assume that Λ_ℓ is of Type 3 or 4. Let $0 \leq \gamma < 1$. If T_i is near a singular corner (i.e., $R_i = 0$), then for any $u_F \in V_\gamma^2(T_i)$, we have

$$\left| u_F - L_h u_F \right|_{1,T_i} \lesssim h^{\frac{1}{\nu_\ell} - \frac{\gamma}{\mu_\ell}} \left| u_F \right|_{2,\gamma,T_i}.$$

Proof
The result follows by mapping T_i to $\hat{T}$ as in Lemma 4.8 of [8], by using the estimate (26), and then mapping back to T_i by using the properties (3.2) and (3.3) in [8] and the fact that $\hat{R} \lesssim h^{-\frac{1}{\mu_\ell}} R_i$ (we refer to [8, p. 538]). $\square$

Remark 4.4
If T_i is isotropic, the previous Lemma is well known and can be found in [28] (see also [25, §8.4]).

Lemma 4.5
Assume that Λ_ℓ is of Type 4. Let T_i be a triangle belonging to $\bar{\Lambda}_\ell \cap F$ such that $R_i > 0$, then

$$R_i \gtrsim h^{\frac{1}{\mu_\ell}}.$$

Proof
Without loss of generality, by a relabeling, we can always assume that T_0 is the triangle that contains the singular vertex v_ℓ. Then, it has two edges that contain v_ℓ; the first one is the edge in the x_1-axis and is of length $\sim h^{\frac{1}{\mu_\ell}}$, while the other one is of length $\sim h^{\frac{1}{\mu_\ell}}$. Moreover, as the angle between these two edges is independent of the mesh, the ball of center v_ℓ and radius $ch^{\frac{1}{\mu_\ell}}$ intersects only T_0 by choosing c small enough. The estimate follows from the definition of the distance. $\qquad\square$

5. Numerical examples

In this section, we present some numerical examples to illustrate the theory presented in the previous sections.

We will solve the boundary value problem (1) using the FEM with linear elements on a polyhedral domain. The domain is given as follows. We let $\tilde{T}$ be the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0.5, 0.5)$, and let the domain be the prism $\Omega := ((0, 1)^2 \setminus \tilde{T}) \times (0, 1)$. We refer to the labeling in Figure 1 in what follows. We will solve (1) in variational form (2) with data $f = 1$ and $g = 0$. The interior angle between the two faces that contain the edge $e := \overline{v_2 v_7}$ is $3\pi/2$. Based on the estimates in (10) and Theorem 2.5, e is the singular edge; and the solution u admits a decomposition into the singular and regular parts with regularity determined by $\lambda_e = 2/3$. By Theorem 2.5, the location of the face F, where the Ventcel boundary condition is imposed does not drastically affect the regularity of the solution.

To verify our theory, we implement two sets of numerical tests regarding different locations of the special boundary face $F = \Gamma_V$: (i) F is the bottom face of prism Ω, with vertices v_1, v_2, v_3, v_4, and v_5; (ii) F is a face that contains the singular edge with vertices v_2, v_3, v_7, and v_8.

For both cases, the singular parts u_S of the solution have anisotropic exponents and belong to the same weighted space. Moreover, by Corollary 3.4, it is sufficient to choose the parameters in (14) corresponding to the singular edge such that $\mu_\ell < 2/3$ and $v_\ell = 1$, in order to achieve the optimal (first-order) convergence rate.

In Table I, we list the convergence rates of the numerical solution for the aforementioned model problems with $v_\ell = 1$, but with different values of the mesh grading parameter μ_ℓ. We let N be the number of degrees of freedom in the discrete system. Then, the mesh size satisfies $h \sim N^{-1/3}$. Because the exact solution is not known, the convergence rate is computed using the numerical solutions for successive mesh refinements, u_{2h} u_h, and $u_{h/2}$, as

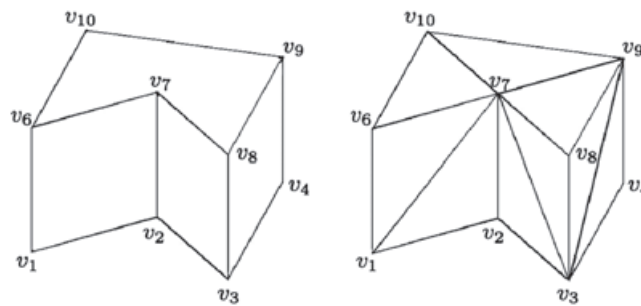$$\text{the convergence rate} = \log_2 \left(\frac{\|u_h - u_{2h}\|_V}{\|u_{h/2} - u_h\|_V} \right), \tag{31}$$

Figure 1. The computational domain Ω (left) and the macro elements (right).

Table I. Convergence rates for different values of μ_ℓ: (i) F is the bottom face (left); (ii) F is a side face containing the singular edge (right).

$h \backslash \mu_\ell$	0.58	0.76	1.00	0.58	0.76	1.00
2^{-3}	0.834	0.843	0.825	0.821	0.833	0.825
2^{-4}	0.938	0.930	0.890	0.936	0.896	0.889
2^{-5}	0.977	0.960	0.894	0.977	0.899	0.890
2^{-6}	0.991	0.968	0.871	0.990	0.876	0.866
2^{-7}	0.995	0.968	0.837	1.000	0.842	0.831

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where u_{2h} and $u_{h/2}$ are the finite element solutions with mesh parameters $2h$ and $h/2$, respectively. Therefore, as h decreases, the asymptotic convergence rate in (31) is a reasonable indicator of the actual convergence rate for the finite element solution.

It is clear from the table that for both cases, the first-order convergence rate is obtained for $\mu_\ell = 0.58 < 2/3$, while we lose the optimal convergence rate if $\mu_\ell = 0.76, 1.00$, both larger than the critical value $2/3$. When $\mu_\ell = 0.76$, that is, $2/3 < \mu_\ell < 1$, this choice still leads to an anisotropic mesh graded toward the singular edge, but the grading is insufficient to resolve the singularity in the solution, and hence does not give rise to the predicted first-order convergence rate. These results are in strong agreement with the theoretical results in Sections 3 and 4.

Acknowledgements

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