

# UNIFORM CONVERGENCE OF THE MULTIGRID $V$ -CYCLE ON GRADED MESHES

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ABSTRACT. We prove the uniform convergence of the multigrid  $V$ -cycle on graded meshes for corner-like singularities of elliptic equations on a bounded domain  $\Omega \subset \mathbb{R}^2$ . In particular, using some weighted Sobolev space  $K_a^m(\Omega)$  and the method of subspace corrections with the elliptic projection decomposition estimate on  $K_a^m(\Omega)$ , we show that the multigrid  $V$ -cycle converges uniformly for piecewise linear functions with standard smoothers (Richardson, weighted Jacobi, Gauss-Seidel, etc.).

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## 1. INTRODUCTION

The multigrid method has proved to be one of the most efficient techniques to solve the large systems of algebraic equations from the finite element discretization of elliptic boundary problems. Many details on the convergence properties of the multigrid method for elliptic equations can be found in monographs and survey papers by Bramble [11], Hackbusch[26], Trottenberg, Oosterlee and Schüller [29], Xu [32] and the references there in.

It is well known that the geometry of the boundary and the change of the boundary condition will influence the regularity of the solution [6, 9, 24, 25]. In particular, if a two-dimensional domain possesses reentrant corners, cracks, or there exist abrupt changes of boundary conditions, the solution of the elliptic boundary problem may have singularities in  $H^2$ . We define the singularities of these types the corner-like singularities, since they can be interpreted as corner singularities (artificial vertices are needed when the boundary condition changes) [28]. Graded meshes [3, 7, 9] are needed to obtain better numerical approximations to the solutions in these cases.

It is non-trivial to analyze the convergence rate of multigrid methods on such graded meshes due to the lack of the regularity of the solution and the non-uniformity of the mesh. One result for the uniform convergence of the multigrid method with the full regularity was given by Braess and Hackbusch [10]; in Brenner's paper [22], the analysis of the convergence rate for partial regularity was presented; Bramble, Pasciak, Wang and Xu [15] developed the convergence estimate without regularity assumptions with the  $L^2$ -projection decomposition. In addition, on graded meshes, based on the approximation property in [7], Yserentant [37] proved the uniform convergence of the multigrid  $W$ -cycle for piecewise linear functions by applying a particular iterative method on each level. There are also classical convergence proofs that use algebraic techniques and derive convergence results based on assumptions related to, but nevertheless different from the regularity of the underlying PDE [17, 30].

In this paper, we shall use the elliptic projection decomposition estimate on the weighted Sobolev space  $K_a^m$  and the method of subspace corrections to prove the uniform convergence of the multigrid  $V$ -cycle with standard subspace smoothers (Richardson, weighted Jacobi, Gauss-Seidel, etc.). To date, this approach has been known to work only for problems with full elliptic regularity. The reason we are able to obtain the uniform convergence result in cases of less regular solutions is that we use special graded meshes, tuned up to capture the correct behavior of the solutions near singularities.

Let  $\Omega$  be a bounded polygonal domain or a domain with cracks in  $\mathbb{R}^2$ . We consider the following elliptic equation with mixed boundary conditions

$$(1) \quad \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega_D \\ \partial u / \partial n = 0 & \text{on } \partial\Omega_N \end{cases}$$

as a prototype problem, where  $\Omega_D$  and  $\Omega_N$  are composed of segments of the boundary. Hence, possible corner-like singularities may appear in the solution. Denote by  $H_D^1(\Omega) = \{u \in H^1(\Omega) \mid u = 0 \text{ on } \partial\Omega_D\}$  the space of all the functions in  $H^1(\Omega)$  with trace 0 on  $\partial\Omega_D$ . Let  $\Gamma_j$ ,  $0 \leq j \leq J$ , be a sequence of appropriately graded triangulations with triangles, which are nested on  $\Omega$ . Denote by  $\mathcal{M}_j$ ,  $0 \leq j \leq J$ , the finite element space associated to the linear Lagrange triangle [23]. Then,

$$\mathcal{M}_0 \subset \mathcal{M}_1 \subset \dots \subset \mathcal{M}_j \subset \dots \subset \mathcal{M}_J \subset H_D^1(\Omega).$$

We solve Equation (1) by looking for an approximation  $u_J \in \mathcal{M}_J$ , such that

$$a(u_J, v_J) = (Au_J, v_J) = (\nabla u_J, \nabla v_J) = \langle f, v_J \rangle, \quad \forall v_J \in \mathcal{M}_J, \quad f \in (H_D^1(\Omega))'$$

where  $A$  is the elliptic operator and  $A = -\Delta$  for Equation (1). Let  $N_J$  be the dimension of the space  $\mathcal{M}_J$ . The following quasi-optimal rate of convergence for the finite element approximation  $u_J \in \mathcal{M}_J$  can be recovered on  $\Gamma_J$ ,

$$\|u - u_J\|_{H^1(\Omega)} \leq CN_J^{-1/2} \|f\|_{L^2(\Omega)}.$$

To be more precise, let  $n$  be the number of iterations on each level. The main objective of this paper is to prove the uniform convergence of the multigrid  $V$ -cycle with standard subspace smoothers (Richardson, weighted Jacobi, Gauss-Seidel, etc.) for piecewise linear functions on graded meshes. Moreover, we shall show that the convergence rate  $c$  satisfies

$$c \leq \frac{c_1}{c_1 + c_2 n},$$

where  $c_1$  and  $c_2$  are constants related to the elliptic equation and the smoother, independent of the mesh size. This will also provide a method to estimate the efficiency of other subspace smoothers on graded meshes.

The rest of this paper is organized as follows. In Section 2, we shall introduce the weighted Sobolev space  $K_a^m(\Omega)$  for the boundary value problem (1) and the method of subspace corrections. We shall briefly describe the generation of the graded mesh on which the finite element solution converges to the exact solution of (1) quasi-optimally. In Section 3, the approximation and smoothing properties are followed by our main theorem.

## 2. WEIGHTED SOBOLEV SPACES AND THE METHOD OF SUBSPACE CORRECTIONS

In this section, we shall first introduce the weighted Sobolev space  $K_a^m(\Omega)$  and the mesh refinements to recover the quasi-optimal rates of convergence of the finite element solution. Then, we shall describe the method of subspace corrections and the technique for estimating the norm of the product of non-expensive operators.

**2.1. Weighted Sobolev spaces and graded meshes.** Let  $(x, y) \in \Omega$  be an arbitrary point and  $S = \{S_i\}$  be the set of (artificial) vertices of  $\Omega$ , on which the solution has singularities in  $H^2(\Omega)$ . Denote by  $r_i(x, y)$  the distance from  $(x, y)$  to the vertex  $S_i \in S$ . Let  $\rho(x, y) = \prod_i r_i$  be the smooth function on  $\Omega$ . Then, the weighted Sobolev space  $K_a^m(\Omega)$ ,  $m \geq 0$ , is defined as follows [9, 27]

$$K_a^m(\Omega) = \{u \in H_{loc}^m(\Omega) \mid \rho^{i+j-a} \partial_x^i \partial_y^j u \in L^2(\Omega), i+j \leq m\}.$$

The  $K_a^m$ -norm and seminorm for any function  $v \in K_a^m(\Omega)$  are

$$\begin{aligned} \|v\|_{K_a^m(\Omega)}^2 &:= \sum_{i+j \leq m} \|\rho^{i+j-a} \partial_x^i \partial_y^j v\|_{L^2(\Omega)}^2 \\ |v|_{K_a^m(\Omega)}^2 &:= \sum_{i+j=m} \|\rho^{m-a} \partial_x^i \partial_y^j v\|_{L^2(\Omega)}^2. \end{aligned}$$

Note that  $\rho$  behaves like the distance function  $r_i(x, y)$  near the vertex  $S_i$ . Thus, we have the following proposition and mesh refinements as in [9, 28].

**Proposition 2.1.** *We have  $|v|_{K_1^1(\Omega)} \approx |v|_{H^1(\Omega)}$ ,  $\|v\|_{K_1^0(\Omega)} \geq C\|v\|_{L^2(\Omega)}$  and the Poincare type inequality  $\|v\|_{K_1^0(\Omega)} \leq C|v|_{K_1^1(\Omega)}$  for  $v \in K_1^1(\Omega) \cap \{v|_{\partial\Omega_D} = 0\}$ .*

By  $a \approx b$ , we mean that there are positive constants  $C_1, C_2$ , such that  $C_1 b \leq a \leq C_2 b$ .

Let  $\kappa$  be the ratio of decay of triangles near the set  $S$ . Then, one can choose  $\kappa = 2^{-1/\epsilon}$ , for  $\forall \epsilon < \min(\pi/\alpha_i)$ , where  $\alpha_i$  is the interior angle of vertex  $S_i$ , and  $\alpha_i = 2\pi$  on the artificial vertices where  $\partial\Omega_D$  and  $\partial\Omega_N$  meet. We assume that no triangle consists of more than one point in  $S$  and any  $S_i$  is a vertex of some triangle in the initial triangulation. Let  $\Gamma_i = \{T_k\}$  be the triangulation after  $i$  refinements. Then, for the  $i+1$ th refinement, if the function  $\rho$  is bounded away from 0 on a triangle, new triangles are generated by connecting the mid-points of the old triangle. However, if  $S_i$  is one of the vertices of triangle  $\triangle S_i BC$ , we pick a point  $D$  on  $\overline{S_i B}$  and another point  $E$  on  $\overline{S_i C}$ , such that the following holds for the ratios of the lengths

$$\kappa = \overline{S_i D} / \overline{S_i B} = \overline{S_i E} / \overline{S_i C}.$$

Then, triangle  $\triangle S_i BC$  is divided into four triangles by connecting  $D, E$ , and the mid-point of  $\overline{BC}$ . (Fig. 2.1 and 2.2)

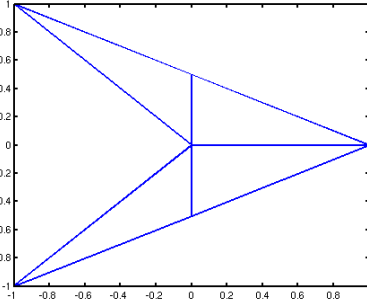


Fig. 2.1. Initial mesh

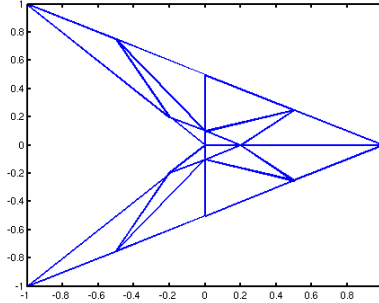


Fig. 2.2. Triangulation after one refinement,  $\kappa = 0.2$

We note that other refinements in [3, 7] also satisfy this condition although they follow different constructions. We now conclude this subsection by restating the following theorem derived in [9, 28].

**Theorem 2.2.** *Let  $u_i \in \mathcal{M}_i$  be the finite element solution of Equation (1). There exists a constant  $B_1 = B_1(\Omega, \kappa, \epsilon)$ , such that*

$$\|u - u_i\|_{H^1(\Omega)} \leq B_1 \dim(\mathcal{M}_i)^{-1/2} \|f\|_{K_{\epsilon-1}^0(\Omega)} \leq B_1 \dim(\mathcal{M}_i)^{-1/2} \|f\|_{L^2(\Omega)},$$

for  $\forall f \in K_{\epsilon-1}^0(\Omega)$ , where  $\mathcal{M}_i$  is the finite element space of linear functions on the graded mesh  $\Gamma_i$  as described in Introduction.

**2.2. The method of subspace corrections.** Let  $H_D^1(\Omega) = \{u \in H^1(\Omega) \mid u = 0 \text{ on } \partial\Omega_D\}$  be the Hilbert space for Equation (1). Recall the graded triangulation  $\Gamma_j$  in the last subsection and the finite element space  $\mathcal{M}_j \in H_D^1(\Omega)$  of piecewise linear functions on  $\Gamma_j$ . In addition, since the meshes are nested, we have

$$\mathcal{M}_0 \subset \mathcal{M}_1 \subset \dots \subset \mathcal{M}_j \subset \dots \subset \mathcal{M}_J \subset H_D^1(\Omega).$$

Let  $A : H_D^1(\Omega) \rightarrow (H_D^1(\Omega))'$  be the differential operator for Equation (1). Then the weak form of Equation (1) is

$$a(u, v_i) = (Au, v_i) = (-\Delta u, v_i) = (\nabla u, \nabla v_i) = \langle f, v_i \rangle, \quad \forall v_i \in \mathcal{M}_i,$$

where the pairing  $(\cdot, \cdot)$  is the inner product in  $L^2(\Omega)$ . In addition,  $a(\cdot, \cdot)$  is a continuous bilinear form on  $H_D^1(\Omega) \times H_D^1(\Omega)$  and also coercive by the Poincare inequality.

Meanwhile, let  $Q_i, P_i : H_D^1(\Omega) \rightarrow \mathcal{M}_i$ ,  $A_i : \mathcal{M}_i \rightarrow \mathcal{M}_i$  be the orthogonal projections and the restriction of  $A$  on  $\mathcal{M}_i$ , respectively by:

$$\begin{aligned} (Q_i u, v_i) &= (u, v_i), \quad \forall u \in H_D^1(\Omega), \forall v_i, v_i \in \mathcal{M}_i, \\ a(P_i u, v_i) &= a(u_i, v_i), \quad (A_i u_i, v_i) = (A u_i, v_i). \end{aligned}$$

Let  $\mathcal{N}_i = \{x_j^i\}$  be the set of node points in  $\Gamma_i$  and  $\phi_k(x_j^i) = \delta_{j,k}$  be the linear finite element nodal basis function corresponding to node  $x_k^i$ . Then the  $i$ th level finite element discretization reads as: Find  $u_i \in \mathcal{M}_i$ , such that

$$(2) \quad A_i u_i = f_i,$$

where  $f_i \in \mathcal{M}_i$  satisfying  $(f_i, v_i) = \langle f, v_i \rangle, \forall v_i \in \mathcal{M}_i$ .

The standard multigrid backslash cycle algorithm solves (2) by the iterative method

$$u_i^l = u_i^{l-1} + B_i(f_i - A_i u_i^{l-1}).$$

The operator  $B_i : \mathcal{M}_i \rightarrow \mathcal{M}_i$ ,  $0 \leq i \leq J$  is recursively defined as follows [33].

**Algorithm 2.1.** Let  $R_i \approx A_i^{-1}$ ,  $i > 0$ , denote a local relaxation method. For  $i = 0$ , define  $B_0 = A_0^{-1}$ . Assume that  $B_{i-1} : \mathcal{M}_{i-1} \rightarrow \mathcal{M}_{i-1}$  is defined. Then,

1. Fine grid smoothing: For  $u_i^0 = 0$  and  $k = 1, 2, \dots, n$

$$u_i^k = u_i^{k-1} + R_i(f_i - A_i u_i^{k-1})$$

2. Coarse grid correction: Find the corrector  $e_{i-1} \in \mathcal{M}_{i-1}$  by the iterator  $B_{i-1}$

$$e_{i-1} = B_{i-1} Q_{i-1}(f_i - A_i u_i).$$

Then,  $B_i f_i = u_i^n + e_{i-1}$ .

In addition, Let  $B_J^v$  be the corresponding operator defined for the multigrid  $V$ -cycle. Then, it satisfies  $I - B_J^v A_J = (I - B_J A_J)^*(I - B_J A_J)$  [33]. With the above algorithm, we have

$$\begin{aligned} (I - B_J A_J)u &= u - u_J^n - e_{J-1} = (I - T_J)u - e_{J-1} \\ &= (I - B_{J-1} A_{J-1})(I - T_J)u, \end{aligned}$$

where  $T_j$  is a linear operator and  $T_i = R_i A_i P_i$  and  $T_0 = P_0$  for  $n = 1$ . A recursive application of the above identity yields

$$(I - B_J A_J) = (I - T_0)(I - T_1) \dots (I - T_J).$$

Define  $\|u\|_a^2 = a(u, u) = (Au, u)$  on  $\Omega$ . Then, for the uniform convergence of the multigrid  $V$ -cycle, we need to show that

$$\|I - B_J A_J\|_a^2 \leq c < 1,$$

where  $c$  is independent of  $J$ .

Associated with each  $T_i$ , we introduce its symmetrization

$$\bar{T}_i = T_i + T_i^* - T_i^* T_i,$$

where  $T_i^*$  is the adjoint operator of  $T_i$  with respect to the inner product  $a(\cdot, \cdot)$ . By a well-known result of [34], we have the following estimate

$$\|I - B_J A_J\|_a^2 = \frac{c_0}{1 + c_0},$$

where

$$c_0 \leq \sup_{\|v\|_a=1} \sum_{i=1}^J a((\bar{T}_i^{-1} - I)(P_i - P_{i-1})v, (P_i - P_{i-1})v).$$

From this starting point to prove the uniform convergence, we will concentrate on the estimate on the constant  $c_0$ . One may notice that the above presentation is in terms of operators, while the matrix representation of the iteration is often used in practice. We conclude this section by providing the relation between the operator representation and the matrix representation.

**Lemma 2.3.** *If  $R_D$  is the corresponding matrix representation of the subspace smoother  $R : \mathcal{M}_i \rightarrow \mathcal{M}_i$  and  $A_D$  is the matrix representation of  $A$  on  $\mathcal{M}_i$ , then the following identities hold for  $u \in \mathcal{M}_i$  and  $f \in \mathcal{M}_i$*

$$\begin{aligned} R(f) &= \sum_j \left( \sum_k (R_D)_{j,k}(f, \phi_k) \right) \phi_j \\ R(Au) &= \sum_j \left( \sum_k (R_D)_{j,k}(A_D u_D^l)_k \right) \phi_j, \end{aligned}$$

where  $(R_D)_{j,k}$  represents the common element of the  $j$ th row and the  $k$ th column of the matrix.

*Proof.* The subspace correction method in terms of operators on a certain level can be written in the following way. Given the number of iterations  $n$ , then for  $u^l \in \mathcal{M}_i$  and  $l < n$ ,

$$u^l = u^{l-1} + R(f - Au^{l-1}).$$

Then, the inner product with  $\phi_j$  leads to

$$(u^l, \phi_j) = (u^{l-1}, \phi_j) + (R(f), \phi_j) - (R(Au^{l-1}), \phi_j).$$

By comparison, the corresponding iteration in the matrix representation is

$$u_D^l = u_D^{l-1} + R_D(f_D - A_D u_D),$$

with  $u^l = \sum_j (u_D^l)_j \phi_j$ ,  $(f_D)_j = (f, \phi_j)$  and  $(A_D)_{j,k} = a(\phi_j, \phi_k)$ . For a better presentation, we introduce the mass matrix  $M$ , such that  $M_{j,k} = (\phi_j, \phi_k)$ . Thus, we have the following relations between the functions  $u^l$ ,  $u^{l-1}$  and the vectors  $u_D^l$ ,  $u_D^{l-1}$ ,

$$(u^l, \phi_j) = (M u_D^l)_j, \quad (u^{l-1}, \phi_j) = (M u_D^{l-1})_j.$$

Therefore, taking the matrix representation of the iteration into account, since they are equivalent, we have

$$(R(f), \phi_j) = (M R_D f_D)_j, \quad (R(Au^{l-1}), \phi_j) = (M R_D A_D u_D)_j.$$

Based on the definition of  $f_D$  and  $M$ , the linear operator  $R$  is defined as follows,

$$R(f) = \sum_j \left( \sum_k (R_D)_{j,k}(f, \phi_k) \right) \phi_j.$$

As for  $R(Au^l)$ , since  $(Au^l, \phi_j) = a(u^l, \phi_j) = (A_D u_D^l)_j$ ,

$$\begin{aligned} R(Au^l) &= \sum_j \left( \sum_k (R_D)_{j,k} (Au^l, \phi_k) \right) \phi_j \\ &= \sum_j \left( \sum_k (R_D)_{j,k} (A_D u_D^l)_k \right) \phi_j, \end{aligned}$$

which completes the proof.  $\square$

### 3. UNIFORM CONVERGENCE OF THE MULTIGRID METHOD ON GRADED MESHES

We now are going to estimate the constant  $c_0$  introduced in Section 2 and to give the main theorem in this paper. For the proof of the uniform convergence, we shall start with some lemmas first. To better explain our results, we assume there is only one vertex  $S_0$  of  $\Omega$ , on which the solution of Equation (1) has a singularity in  $H^2(\Omega)$ . The same argument, however, will work on domains with multiple singular vertices. Recall the way we refine the mesh in Section 2. Denote by  $T_i^{S_0}$  the initial triangles with vertex  $S_0$ . Thus, the mesh generation contains the following. After  $N$  refinements,  $\cup T_i^{S_0}$  is chopped into  $N + 1$  sub-domains  $D_n$ ,  $0 \leq n \leq N$ , such that  $\rho(x, y) \approx \kappa^n$  on  $D_n$  for  $0 \leq n < N$  and  $\rho(x, y) \leq C\kappa^N$  on  $D_N$ . Then, sub-triangles are generated in these layers and the mesh size on  $D_n$  is  $O(\kappa^n 2^{n-N})$ , for  $0 \leq n \leq N$ . (Fig. 3.1 and 3.2)

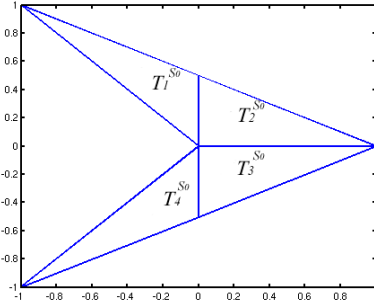


Fig. 3.1. Initial triangles with vertex  $S_0$

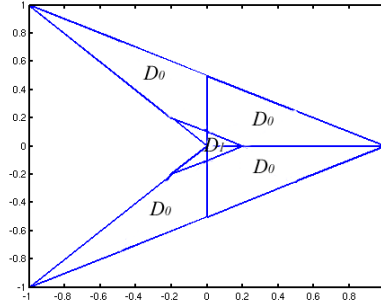


Fig. 3.2. Layer  $D_0$  and  $D_1$  after one refinement,  $\kappa = 0.2$

Note that  $\Omega = (\cup D_n) \cup (\Omega \setminus \cup D_n)$ . Let  $\partial D_n$  be the boundary of  $D_n$ . Then, we define a piecewise constant function  $r_p(x, y)$  on  $\bar{\Omega}$  as follows. Let the restriction of  $r_p$  on every  $T_i^{S_0} \cap D_n$  be a constant, such that

$$r_p(x, y) = \begin{cases} (1/2\kappa)^n & \text{on } \bar{D}_n \setminus \partial D_{n-1}, \quad \text{for } 1 < n \leq N \\ 1 & \text{otherwise,} \end{cases}$$

where  $N$  is the number of refinements. Thus  $N = 1$  for  $\Gamma_1$ ,  $N = 2$  for  $\Gamma_2$ , and  $N = i$  for  $\Gamma_i$ . Recall that  $\epsilon < 1$  is the parameter in  $\kappa$ , such that  $\kappa = 2^{-1/\epsilon}$ .

Denote by  $(\cdot, \cdot)_{r_p}$  the weighted inner product with respect to  $r_p$ ,

$$(u, v)_{r_p} = (r_p u, r_p v) = \int_{\Omega} r_p^2 uv.$$

Then, we have the estimate below.

**Lemma 3.1.**

$$(u_i - P_{i-1} u_i, u_i - P_{i-1} u_i)_{r_p} \leq \frac{c_1}{N_i} a(u_i, u_i), \quad \forall u_i \in \mathcal{M}_i.$$

where  $N_i = O(2^{2i})$  is the dimension of  $\mathcal{M}_i$

*Proof.* This lemma can be proved by the duality argument as follows.

Consider the following boundary value problem

$$\begin{cases} -\Delta w = r_p^2(u_i - P_{i-1}u_i) & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega_D \\ \partial w / \partial n = 0 & \text{on } \partial\Omega_N \end{cases}$$

Then, since  $P_{i-1}w \in \mathcal{M}_{i-1}$ , we have

$$\begin{aligned} (r_p(u_i - P_{i-1}u_i), r_p(u_i - P_{i-1}u_i)) &= (r_p^2(u_i - P_{i-1}u_i), u_i - P_{i-1}u_i) \\ &= (\nabla w, \nabla(u_i - P_{i-1}u_i)) \\ &= (\nabla(w - P_{i-1}w), \nabla(u_i - P_{i-1}u_i)). \end{aligned}$$

We note that  $\Delta w$  is a piecewise function on triangulation  $\Gamma_i$  that is derived after  $i$  refinements. Then, based on the arguments in Theorem 2.2, we have

$$\begin{aligned} |w - P_{i-1}w|_{H^1(\Omega)}^2 &\leq (C_1/N_{i-1}) \|\Delta w\|_{K_{\epsilon-1}^0(\Omega)}^2 \\ &= (C_1/N_{i-1}) \left( \sum_{n=0}^i \|\rho^{1-\epsilon} \Delta w\|_{L^2(D_n)}^2 + \|\rho^{1-\epsilon} \Delta w\|_{L^2(\Omega \setminus \cup D_n)}^2 \right) \\ &\leq (C/N_{i-1}) \left( \sum_{n=0}^i \|\kappa^{n(1-\epsilon)} \Delta w\|_{L^2(D_n)}^2 + \|\Delta w\|_{L^2(\Omega \setminus \cup D_n)}^2 \right) \\ &= (C/N_{i-1}) \left( \sum_{n=0}^i \|2^n 2^{-\frac{n}{\epsilon}} \Delta w\|_{L^2(D_n)}^2 + \|\Delta w\|_{L^2(\Omega \setminus \cup D_n)}^2 \right) \\ &= (C/N_{i-1}) \left( \sum_{n=0}^i \|2^n \kappa^n \Delta w\|_{L^2(D_n)}^2 + \|\Delta w\|_{L^2(\Omega \setminus \cup D_n)}^2 \right) \\ &= (C/N_{i-1}) \left( \sum_{n=0}^i \|r_p^{-1} \Delta w\|_{L^2(D_n)}^2 + \|\Delta w\|_{L^2(\Omega \setminus \cup D_n)}^2 \right) = (C/N_{i-1}) \|r_p^{-1} \Delta w\|_{L^2(\Omega)}^2. \end{aligned}$$

Note that  $N_i = O(N_{i-1})$ . Therefore, we have the following estimates to complete the proof,

$$\begin{aligned} \|u_i - P_{i-1}u_i\|_{r_p}^2 &\leq \frac{|w - P_{i-1}w|_{H^1}^2 |u_i - P_{i-1}u_i|_{H^1}^2}{\|(u_i - P_{i-1}u_i)\|_{r_p}^2} \\ &= \frac{|w - P_{i-1}w|_{H^1}^2 |u_i - P_{i-1}u_i|_{H^1}^2}{\|r_p^{-1} \Delta w\|_{L^2}^2} \\ &\leq \frac{c_1}{N_i} |u_i - P_{i-1}u_i|_{H^1}^2 \leq \frac{c_1}{N_i} a(u_i, u_i). \end{aligned}$$

□

We here state another lemma regarding the smoother  $\bar{R}_i$  on  $\mathcal{M}_i$ , where  $\bar{R}_i$  is the symmetrization of  $R_i$ ,  $\bar{R}_i = R_i + R_i^t - R_i^t A_i R_i$ .

**Lemma 3.2.** *For any subspace smoother  $\bar{R}_i : \mathcal{M}_i \rightarrow \mathcal{M}_i$ , assume its corresponding matrix representation  $\bar{R}_D$  satisfies*

$$V_D^t \bar{R}_D V_D \geq C V_D^t V_D, \quad \forall V_D \in \mathbb{R}^k, \quad C > 0,$$

on every level  $i$ , where  $C$  is a constant independent of  $i$ , and  $k$  is the dimension of  $\bar{R}_D$ . Then, the following estimate holds for the graded mesh  $\Gamma_i$ ,

$$\frac{c_2}{N_i} (\bar{R}_i v, v) \leq (\bar{R}_i v, \bar{R}_i v)_{r_p}, \quad \forall v \in \mathcal{M}_i,$$

where  $N_i$  is the the dimension of  $\mathcal{M}_i$  as in Lemma 3.1.

*Proof.* For any  $v = \sum_j (V_D)_j \phi_j \in \mathcal{M}_i$ , from Lemma 2.3, we have

$$\begin{aligned} (\bar{R}_i v, v) &= \left( \sum_k (\bar{R}_D M V_D)_k \phi_k, \sum_j (V_D)_j \phi_j \right) = V_D^T M \bar{R}_D M V_D. \\ (\bar{R}_i v, \bar{R}_i v)_{r_p} &= \left( \sum_j (\bar{R}_D M V_D)_j r_p \phi_j, \sum_k (\bar{R}_D M V_D)_k r_p \phi_k \right) \\ &= V_D^T M \bar{R}_D \tilde{M} \bar{R}_D M V_D, \end{aligned}$$

where  $M$  is the mass matrix and  $(\tilde{M})_{j,k} = (r_p \phi_j, r_p \phi_k)$ . Note that both  $M$  and  $\tilde{M}$  are symmetric positive definite (SPD). Hence,  $\tilde{M} \approx 1/N_i$ , since the mesh size at the support of  $\phi_j$  is  $O(\kappa^n 2^{n-i})$  and  $r_p = (1/2\kappa)^n$  if the support is sitting in  $D_n$ . Thus, by setting  $\xi = \bar{R}_D^{1/2} M V_D$ , to complete the proof, it is equivalent to show that

$$\xi^T \bar{R}_D^{1/2} \tilde{M} \bar{R}_D^{1/2} \xi \geq (C_1/N_i) \xi^T \xi.$$

From our estimates on  $\tilde{M}$ , we have

$$\xi^T \bar{R}_D^{1/2} \tilde{M} \bar{R}_D^{1/2} \xi \approx (1/N_i) \xi^T \bar{R}_D \xi \geq (C/N_i) \xi^T \xi.$$

□

*Remark 3.3.* Recall the definition of the graded mesh. The triangles are shape-regular elements and the minimum angle of the triangles are bounded from 0. Therefore,  $(A_D)_{i,j} = O(1)$  and the maximum eigenvalue of  $A_D$  is bounded. In fact, one can verify that standard smoothers (Richardson, weighted Jacobi, Gauss-Seidel, etc.) satisfy Lemma 3.2, since  $(R_D)_{i,j} = O(1)$ . Moreover, if  $R_D$  is SPD and the spectral radius  $\rho(R_D A_D) \leq \omega$ , for  $0 < \omega < 2$ , then  $a(R_i A_i v, v) = (A_i R_i A_i v, v) = V_D^T A_D R_D A_D V_D < \omega a(v, v)$ . The last identity is from the similarity of the matrix  $A_D^{1/2} R_D A_D^{1/2}$  and the matrix  $R_D A_D$ . Hence,  $a(R_i A_i v, v) \leq \omega a(v, v)$  for standard smoothers.

Before we state our main theorem, we shall introduce some notation that will be used in the proof. Recall the operator  $T_i$ . Denote by  $R_i$  a subspace smoother satisfy Lemma 3.2 and the spectral radius  $\rho(R_i A_i) \leq \omega$  for  $0 < \omega < 2$ . Let  $R_i^t$  be the adjoint of  $R_i$  with respect to  $(\cdot, \cdot)$ . We can apply  $R_i$  and  $R_i^t$  alternatively with  $n$  smoothings on the  $i$ th level. We also define the operator  $G_i$  and  $G_i^*$  as follows,

$$G_i = I - R_i A_i, \quad G_i^* = I - R_i^t A_i.$$

With this type of subspace correction, we have

$$T_i = \begin{cases} P_i - (G_i^* G_i)^{\frac{n}{2}} P_i & \text{for even } n \\ P_i - G_i (G_i^* G_i)^{\frac{n-1}{2}} P_i & \text{for odd } n \end{cases}$$

Therefore, if we define

$$G_{i,n} = \begin{cases} G_i^* G_i & \text{for even } n \\ G_i G_i^* & \text{for odd } n, \end{cases}$$

since  $P_i^2 = P_i$ , the presentation for  $\bar{T}_i$  is

$$\bar{T}_i = T_i + T_i^* - T_i^* T_i = (I - G_{i,n}^n) P_i.$$

With this setting-up, the uniform convergence of the multigrid method can be proved as follows.



**Theorem 3.4.** *On every triangulation  $\Gamma_i$ , suppose that the smoother on each subspace  $\mathcal{M}_i$  satisfies Lemma 3.2. Then we have  $\|I - B_J A_J\|_a^2 = \frac{c_0}{1+c_0} \leq \frac{c_1}{c_1+c_2n}$ .*

*Proof.* To estimate the constant  $c_0$ , we consider the decomposition  $v = \sum_i v_i$  for any  $v \in \mathcal{M}_J$  with

$$v_i = (P_i - P_{i-1})v.$$

Since  $P_{i-1} = P_{i-1}P_i = P_iP_{i-1}$ , Lemma 3.1 implies that

$$(3) \quad N_i(v_i, v_i)_{r_p} \leq c_1 a(v_i, v_i).$$

By the identity of Xu and Zikatanov [34] and the conditions in Lemma 3.2, we have

$$\begin{aligned} a(\bar{T}_i^{-1}(I - \bar{T}_i)v_i, v_i) &= a((I - G_{i,n}^n)^{-1}G_{i,n}^n v_i, v_i) \\ &= (\bar{R}_i^{-1}\bar{R}_i A_i (I - G_{i,n}^n)^{-1}G_{i,n}^n v_i, v_i) \\ &= (\bar{R}_i^{-1}(I - G_{i,n})(I - G_{i,n}^n)^{-1}G_{i,n}^n v_i, v_i) \\ &\leq \max_{t \in [0,1]} (I - t)(I - t^n)^{-1}t^n (\bar{R}_i^{-1}v_i, v_i) \\ &\leq \frac{1}{n} (\bar{R}_i^{-1}v_i, v_i) \leq \frac{N_i}{c_2 n} (v_i, v_i)_{r_p}. \end{aligned}$$

On the other hand, from the relation in (3),

$$\sum_{i=0}^J a(\bar{T}_i^{-1}(I - \bar{T}_i)v_i, v_i) \leq \sum_{i=1}^J \frac{N_i}{c_2 n} (v_i, v_i)_{r_p} \leq \sum_{i=0}^J \frac{c_1}{c_2 n} a(v_i, v_i) = \frac{c_1}{c_2 n} a(v, v).$$

Therefore,  $c_0 \leq \frac{c_1}{c_2 n}$  and consequently, the method of subspace corrections has the following convergence estimate for the multigrid  $V$ -cycle:

$$\|I - B_J A_J\|_a^2 = \frac{c_0}{1+c_0} \leq \frac{c_1}{c_1+c_2n},$$

which completes the proof.  $\square$

*Acknowledgements:* We would like to thank Long Chen, Victor Nistor, Jinchao Xu and Ludmil Zikatanov for useful suggestions and discussions during the preparation of this manuscript.

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