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A note on the conditioning of a class of generalized finite element methods

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Dedicated to George Hsiao on the occasion of his 75th birthday

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ABSTRACT

We study the asymptotic behavior of the condition number of the linear system from the discretization of a class of generalized finite element methods for solving second-order elliptic boundary value problems. Allowing local approximation spaces with polynomials of different degrees and different local patch sizes (local refinements), we give bounds on the condition number in relation to the patch size and the dimension of the global approximation space in which the shape functions are in general not polynomials. Numerical tests verify the theorems.

1. Introduction

As a meshless method to approximate the solution of the boundary value problem, the generalized finite element method (GFEM) has been increasingly popular in practical computations of science and engineering in the past two decades. Based on the partition of unity method, this method originated in the work of Babuška et al. [4] and has been further studied and developed under different names by many people (see [3,5,21,20,11–13,16–19,25,26] and references therein).

With the flexibility on the selection of the local approximation spaces and of the partition of unity functions, the GFEM can overcome some major difficulties in the mesh generation on complex domains [25,26]; it is also possible to incorporate certain a priori understandings of the solution in the numerical method [23]. For example, in the presence of corner singularities in the solution, one may add the known singular expression into the approximation space to improve the accuracy of the numerical solution. In addition, by carefully designing the partition of unity function, one may construct a highly smooth approximation space for solving high-order equations and some time-related problems.

One of the major issues on the implementation of the GFEM is the estimate and development of effective solvers for the linear system from the numerical discretization. Additional difficulties in developing such solvers come from the flexibility in the construction of the approximation space and the possible existence of non-polynomial approximation functions. See [10,15,30] for example. The convergence property of many iterative methods (e.g., conjugate gradient methods and multigrid methods) is often related to the condition number of the underlying linear system. Therefore, the study on the condition number of the linear system from the GFEM is of both theoretical interest and practical importance. In particular, see [10] for the analysis of multilevel Schwarz preconditioners for a class of partition of unity methods that lead to uniformly bounded condition numbers.

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In this paper, we consider the conditioning of a class of GFEMs solving second-order elliptic boundary value problems on a bounded domain $\Omega \subset \mathbb{R}^n$. These GFEMs utilize the "flap-top" partition of unity functions and enrich the local approximation space with polynomials. This approach has shown its advantage in providing good numerical approximations (Theorem 2.3) and avoiding singular linear systems by guaranteeing the linear independence of the functions in the approximation space (Proposition 2.7).

In the analysis on the condition number of the resulting system, we particularly take into account the flexibility on the selection of different parameters (e.g., partition of unity functions, the geometry of patches, and local approximation spaces) in the GFEM. Precisely, we allow the use of polynomials of different degrees in the local approximation space on different patches, the use of different shapes for the local patches, and the use of possible special local refinements (different patch sizes). In contrast to the usual finite element method, the shape functions in the GFEM, i.e., the products of a local approximation basis function and the partition of unity function, are in general not polynomials and the patch may not be a simplex. This disqualifies certain arguments in the standard finite element theory [8,9,28] and raises difficulties in the analysis. Our main results (Theorem 3.10) provide upper bounds for the growth rate of the condition number of the scaled stiffness matrix, as the dimension of the approximation space increases. These bounds also depend on the dimension n and on the ratio of the largest to the smallest size of the patches. On the other hand, the growth rate of the condition number of the mass matrix (Theorem 3.5) only depends on the ratio of the largest and smallest sizes of the patches and is not affected by the dimension of the approximation space. Note that these growth rates turn out to resemble those in the usual finite element method [7], despite these flexibilities on the construction of the GFEM. Different from the special preconditioning techniques proposed in [10,30], in this paper, we focus on the conditioning of the original matrix system from a class of GFEMs, from which further multilevel preconditioners (e.g., multigrid methods and Schwarz preconditioners) may be developed for these GFEMs.

Another practical concern on these GFEMs is the magnitude of the condition number. It has been observed in practice that a bad choice of the partition of unity functions and the geometry of the patches can be the dominating factor that severely increases the magnitude of the condition number and therefore, worsens the conditioning of the system. The effect of the choice of these parameters on the condition number can be potentially investigated by further analyzing the constants in our estimates, and in turn improve the effectiveness of the GFEMs.

The rest of this paper is organized as follows. In Section 2, we briefly describe the GFEM for solving elliptic partial differential equations. Some existing results from the literature and the notation that will be used throughout the text are summarized. In Section 3, we show our main results for the estimate of the growth rate of the condition number in detail. In Section 4, we present numerical results for a model problem on different domains. In each case, the growth rates of the condition number are in complete agreement with our theoretical prediction. These results also indicate some relations between the partition of unity and the magnitude of the condition number, which is a future direction we would like to pursue.

2. Preliminaries and notation

In this section, we briefly describe the generalized finite element method for solving second-order elliptic boundary value problems and give basic assumptions on a class of GFEMs that we shall analyze in Section 3.

2.1. The GFEM

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. A second-order elliptic boundary value problem on Ω is often solved variationally by incorporating the boundary conditions (Dirichlet, Neumann, etc.) in a closed subspace $V \subset H^1(\Omega)$. This process results in a weak formulation, i.e., finding $u \in V$, such that

$$a(u,v) = (f,v), \quad \forall v \in V, \tag{1}$$

where $a(\cdot,\cdot)$ is the bilinear form associated to the original equation. We further suppose $a(\cdot,\cdot)$ is symmetric and is both continuous and coercive on V. Namely, there exist constants $\alpha_0, \alpha_1 > 0$, such that

$$a(u,v) \leqslant \alpha_0 \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}, \quad \forall u, v \in V, \tag{2}$$

and

$$a(u,u) \geqslant \alpha_1 \|u\|^2_{H^1(\Omega)} \quad \forall u \in V. \tag{3}$$

The above inequalities lead to a unique solution $u \in V \subset H^1(\Omega)$ of (1) for any $f \in V'$ by the Lax–Milgram Lemma.

For example, consider the following equation with the zero Dirichlet boundary condition

$$\begin{cases} -\operatorname{div} A\nabla u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $A = (a_{i,j}(x))$ is a symmetric matrix, and $a_{i,j}(x)$ are smooth functions satisfying for a constant $C > 0$,

$$\sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \geqslant C\sum_{i=1}^{n}\xi_i^2, \quad \forall \xi \in \mathbb{R}^n,\ \forall x \in \Omega. \tag{4}$$

Then, $V = H_0^1(\Omega)$ and the weak formulation reads

$$a(u,v) = \int_{\Omega} \sum_{i,j=1}^{n} a_{i,j}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,dx = \int_{\Omega} f\,v\,dx = (f,v), \quad \forall v \in H_0^1(\Omega). \tag{5}$$

It is clear that the bilinear form $a(\cdot,\cdot)$ in (5) is continuous on $H_0^1(\Omega)$. Its coercivity can be verified by (4) and the Poincaré inequality.

Definition 2.1. The GFEM is a Galerkin method that approximates the solution of Eq. (1) by using functions in a finite-dimensional subspace. The major components of a GFEM include (see also [1,2] and references therein):

(i) A collection $\{\omega_i\}_{i=1}^{I}$ of small open *sets* (*patches*) that form an open cover of the domain Ω,

$$\omega_i \subset \Omega \quad \text{and} \quad \Omega = \bigcup_{i=1}^{I}\omega_i.$$

In addition, any $x \in \Omega$ belongs to at most $\mathcal{K}$ patches ω_i.

(ii) A family of functions $\{\phi_i\}_{i=1}^{I}$ that form a *partition of unity* subordinate to the finite covering $\{\omega_i\}$ (i.e., $\operatorname{supp}\phi_i \subset \bar{\omega}_i$) satisfying

$$\sum_{i=1}^{I}\phi_i(x) = 1, \quad \forall x \in \Omega,$$
$$\max_{x\in\Omega}\left|\phi_i(x)\right| \leqslant C_0 \quad \text{and} \quad \max_{x\in\Omega}\left|\nabla\phi_i(x)\right| \leqslant C_1/h_i, \tag{6}$$

for constants $C_0, C_1 > 0$, where $h_i := \operatorname{diam}(\omega_i)$ denotes the diameter of the patch ω_i.

(iii) A *local approximation space* V_i defined on the patch ω_i, which is an m_i-dimensional space of functions, i.e., $V_i = \operatorname{span}\{\xi_{i,j}\}$, where $\xi_{i,j}$, $1 \leqslant j \leqslant m_i$, are the basis functions of V_i.

Define the shape function $\eta_{i,j} := \phi_i\xi_{i,j}$. The GFE space is the linear span of the shape functions

$$S_G := \operatorname{span}\{\eta_{i,j},\ i = 1,\ldots,I,\ j = 1,\ldots,m_i\}. \tag{7}$$

Then, the GFEM for Eq. (1) reads

$$\begin{cases} \text{Find } u_G \in S_G \text{ satisfying} \\ a(u_G, v) = (f, v) \quad \forall v \in S_G. \end{cases} \tag{8}$$

Remark 2.2. The idea of the GFEM is to construct a local space V_i on each patch ω_i, in which the solution of Eq. (1) can be approximated well. Then, using the partition of unity $\{\phi_i\}$, we "paste" the local spaces V_i together to form S_G, which will have good global approximation properties.

To be more precise, a result on the approximation property for functions in S_G is given by the following theorem [21,5].

Theorem 2.3. *Suppose the solution u of Eq.* (1) *can be accurately approximated by* $\xi_i^u \in V_i$ *on* ω_i, *such that*

$$\left\|u - \xi_i^u\right\|_{L^2(\omega_i)} \leqslant \epsilon_i^0 \quad \text{and} \quad \left|u - \xi_i^u\right|_{H^1(\omega_i)} \leqslant \epsilon_i^1,$$

where $\epsilon_i^0, \epsilon_i^1 > 0$ *are generally different on different patches. Then, the global approximation* $\xi^u := \sum_{i=1}^{I}\phi_i\xi_i^u \in S_G$ *satisfies*

$$\left\|u - \xi^u\right\|_{L^2(\Omega)} \leqslant \mathcal{K}^{1/2}C_0\left(\sum_{i=1}^{I}\left(\epsilon_i^0\right)^2\right)^{1/2},$$

$$\left\|u - \xi^u\right\|_{H^1(\Omega)} \leqslant (2\mathcal{K})^{1/2}\left(C_1^2\sum_{i=1}^{I}\frac{(\epsilon_i^0)^2}{h_i^2} + C_0^2\sum_{i=1}^{I}\left(\epsilon_i^1\right)^2\right)^{1/2},$$

where $\mathcal{K}$ *is the upper bound of the number of intersecting patches from Definition* 2.1; *and* C_0 *and* C_1 *are constants from* (6).

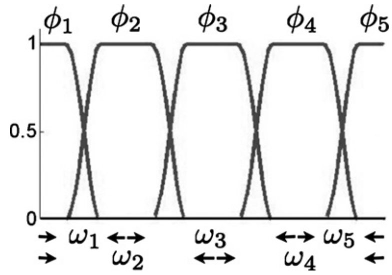

Fig. 1. The open cover $\{\omega_i\}$ and corresponding partition of unity functions ϕ_i for a GFEM in 1-D.

Remark 2.4. Suppose $S_G \subset V$. Then, by Céa's Theorem, the above estimates in turn give an H^1-error estimate for $u - u_G$,

$$\|u - u_G\|_{H^1(\Omega)} \leqslant C\|u - \xi^u\|_{H^1(\Omega)}.$$

Therefore, $\xi^u \in S_G$ plays a similar role in the estimate as the interpolation function in the usual finite element method [8,9]. We also mention that although the space S_G is usually a subspace of $V \subset H^1(\Omega)$, it is not necessarily the case, especially when we are approximating the Dirichlet boundary condition imposed on the solution (see [6]). Note that different selections of the patches ω_i, the partition of unity $\{\phi_i\}$, and the local approximation spaces V_i give rise to different GFEMs. In addition, several classical finite element methods can also be considered as GFEMs [2].

2.2. Notation and assumptions

Recall that the solution of (8) can be written as

$$u_G = \sum_{i=1}^{I} \sum_{j \leqslant m_i} \tilde{v}_{i,j} \eta_{i,j} = \sum_{i=1}^{I} \sum_{j \leqslant m_i} \tilde{v}_{i,j} \phi_i \xi_{i,j} \in S_G, \quad \tilde{v}_{i,j} \in \mathbb{R}.$$

Let $\tilde{\mathbf{V}} := (\tilde{v}_{i,j})$ be the unknown vector. Then, the GFEM yields the linear system

$$\tilde{\mathcal{A}}\tilde{\mathbf{V}} = \tilde{\mathbf{B}}, \tag{9}$$

where the stiffness matrix $\tilde{\mathcal{A}}$ and the vector $\tilde{\mathbf{B}}$ are defined by

$$\tilde{\mathcal{A}} := \big(a(\phi_i \xi_{i,j}, \phi_l \xi_{l,k})\big), \qquad \tilde{\mathbf{B}} := \big((f, \phi_l \xi_{l,k})\big), \quad 1 \leqslant i, l \leqslant I, \tag{10}$$

where $1 \leqslant j \leqslant m_i$ and $1 \leqslant k \leqslant m_l$.

Note that different from the usual finite element method (FEM), the shape functions $\{\eta_{i,j}\}$ may be linearly dependent, which results in a non-trivial kernel for the matrix $\tilde{\mathcal{A}}$. For example, if we choose the basis functions of the usual FEM as the partition of unity functions and restrict the local approximation spaces V_i to be polynomial spaces, the matrix $\tilde{\mathcal{A}}$ is shown to be singular in many cases. This, however, is not the direction we will pursue in this paper. More discussions on singular systems of this type can be found in [27,22].

In this paper, we concentrate on invertible stiffness matrices. Precisely, we shall study the condition numbers of the stiffness matrices from a class of GFEMs with additional assumptions on the patches $\{\omega_i\}$, the partition of unity $\{\phi_i\}$, and on the local approximation spaces V_i. These GFEMs have been widely used to solve practical problems (see [24,2,1] and references therein), in order to avoid costly mesh generation and provide accurate numerical solutions.

Assumption 1. In addition to (i) in Definition 2.1, we assume that every patch ω_i is a Lipschitz domain that contains a ball ω_i^ρ, i.e.,

$$\omega_i^\rho \subset \omega_i, \quad 1 \leqslant i \leqslant I,$$

and there exists a constant $0 < \rho < 1$, for all $1 \leqslant i \leqslant I$, such that

$$\operatorname{diam}\big(\omega_i^\rho\big) \geqslant \rho h_i = \rho \operatorname{diam}(\omega_i).$$

We denote by $x_i^\rho \in \omega_i^\rho$ the center of the ball ω_i^ρ.

Assumption 2. In addition to (ii) in Definition 2.1, we assume

$$\phi_i(x) = 1, \quad \forall x \in \omega_i^\rho.$$

Recall that $\{\phi_i\}$ is a partition of unity. Therefore, for $i \neq j$, $\phi_j(x) = 0$, $\forall x \in \omega_i^\rho$, and we have $\omega_i^\rho \cap \omega_j^\rho = \emptyset$. See Fig. 1 for example.

Assumption 3. Let $\mathcal{P}_k(\omega_i)$ be the space of polynomials of degree k on ω_i. For the local approximation spaces on ω_i, if $\bar{\omega}_i \cap \partial\Omega = \emptyset$, we choose $V_i = \mathcal{P}_{k_i}(\omega_i)$; and let $V_i \subset \mathcal{P}_{k_i}(\omega_i)$ if $\bar{\omega}_i \cap \partial\Omega \neq \emptyset$. This is to take into account possible restrictions when approximating the boundary condition (see [6] for example).

We further set the basis functions $\xi_{i,j}$ of V_i, $1 \leqslant i \leqslant I$, to be of the following form

$$\xi_{i,j} \in \mathcal{B}_i := \left\{ \frac{(x - x_i^\rho)^\alpha}{h_i^{|\alpha|}},\ |\alpha| \leqslant k_i \right\}, \tag{11}$$

where x_i^ρ is the center of the ball ω_i^ρ from Assumption 1 and $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is an n-tuple of non-negative integers, such that for $x \in \mathbb{R}^n$, $x^\alpha := \prod_{i=1}^n x_i^{\alpha_i}$ and $|\alpha| = \sum_{i=1}^n \alpha_i$. It is clear that $V_i = \operatorname{span}\{\xi_{i,j}\} = \mathcal{B}_i$ on ω_i, if $\bar{\omega}_i \cap \partial\Omega = \emptyset$, and $V_i = \operatorname{span}\{\xi_{i,j}\} \subset \mathcal{B}_i$, if $\bar{\omega}_i \cap \partial\Omega \neq \emptyset$, due to possible restrictions discussed above.

Before we give the next assumption, we define the reference patch $\hat{\omega}_i$ of ω_i by

$$\hat{\omega}_i := \left\{ \hat{x} = \left(x - x_i^\rho\right)/h_i,\ \forall x \in \omega_i \right\}.$$

For a function $v(x)$, $\forall x \in \omega_i$, define $\hat{v}(\hat{x}) := v(x)$ on $\hat{\omega}_i$, with $\hat{x} = (x - x_i^\rho)/h_i \in \hat{\omega}_i$. Let $\hat{\omega}_i^\rho := \{(x - x_i^\rho)/h_i,\ \forall x \in \omega_i^\rho\} \subset \hat{\omega}_i$ be the reference ball. Note that the center of $\hat{\omega}_i^\rho$ is at the origin. Two patches ω_i and ω_j will be called *linearly equivalent* if there is an $n \times n$ rotation matrix $\mathbf{R}$ about the origin, such that

$$\hat{\omega}_i = \mathbf{R}(\hat{\omega}_j) \quad \text{and} \quad \hat{\omega}_i^\rho = \mathbf{R}\left(\hat{\omega}_j^\rho\right). \tag{12}$$

Let $S_i := \{\eta_{l,k},\ \operatorname{supp}(\eta_{l,k}) \cap \omega_i \neq \emptyset\}$ be the set of shape functions whose support intersecting ω_i (correspondingly, $\hat{S}_i := \{\hat{\eta}_{l,k},\ \operatorname{supp}(\hat{\eta}_{l,k}) \cap \hat{\omega}_i \neq \emptyset\}$). Then we will call $\{\omega_i, S_i\}$ a paired *data set*. Two data sets $\{\omega_k, S_k\}$ and $\{\omega_l, S_l\}$ are said to belong to the same *class*, if ω_k and ω_l are linearly equivalent and the rotation matrix $\mathbf{R}$ in the form of (12) is such that

$$\operatorname{span}\left\{ (\hat{\eta}_{i,j} \circ \mathbf{R})(\hat{x}_l),\ \hat{\eta}_{i,j} \in \hat{S}_k \right\} = \operatorname{span}\left\{ \hat{\eta}_{m,j}(\hat{x}_l),\ \hat{\eta}_{m,j} \in \hat{S}_l \right\}, \quad \forall \hat{x}_l \in \hat{\omega}_l. \tag{13}$$

Assumption 4. We assume that the number of the classes to which all the data sets $\{\omega_i, S_i\}_{i=1}^I$ belong is independent of the dimension of S_G. In addition, we require that intersecting patches have comparable size. Namely, there is a constant $C > 0$, for all $1 \leqslant i \leqslant I$, such that if $\omega_i \cap \omega_j \neq \emptyset$, then,

$$\max_j(h_i, h_j) \leqslant C \min_j(h_i, h_j), \tag{14}$$

where $h_i = \operatorname{diam}(\omega_i)$.

Remark 2.5. The patches ω_i are often assumed to be convex or star-shaped with respect to a ball, such that polynomials have good local approximation properties on ω_i. This assumption, however, is not required to obtain Theorem 2.3 and the results in this paper. The functions in (11) can be defined in other forms as long as they form a set of basis functions of the local polynomial space. The definition in (11) is, however, convenient to formulate the GFEM [2,1].

Remark 2.6. Based on Assumption 4, all the data sets $\{\omega_i, S_i\}$, $1 \leqslant i \leqslant I$, can be generated by a finite number of sets $\{\omega_l, S_l\}_{l=1}^M$ with dilation and rotation. Since the local approximation spaces V_l may consist of polynomials of different degrees on different patches ω_l, we actually allow polynomials of different degrees in the local approximation spaces in the GFEM. In addition, we do not require global uniform patches. Therefore, we may have different patch sizes in different locations (e.g., special local refinements for singularities), as long as the adjacent patches have a comparable size (14). Note that, however, the variations in the local spaces V_i and in the shapes of patches ω_i are limited, since M does not depend on the dimension of S_G. These assumptions hold in many GFEMs [2,24]. We shall give two specific GFEMs satisfying Assumptions 1–4 in Section 4. We shall use Assumption 4 to control the constants in our analysis.

The partition of unity defined in Assumption 2 is often called the flat-top partition of unity. They are in general not piecewise polynomials, which results in non-polynomial shape functions $\eta_{i,j}$. This construction, however, is widely adopted to avoid the possible linear dependence between the shape functions $\eta_{i,j}$.

Proposition 2.7. *Suppose the partition of unity $\{\phi_i\}$ satisfies Assumption 2 and the local basis functions $\xi_{i,j}$ are given by (11). Then, the set of shape functions $\{\eta_{i,j},\ 1 \leqslant i \leqslant I,\ 1 \leqslant j \leqslant m_i\}$ is linearly independent.*

Proof. Suppose the set is linearly dependent. Then there exist constants $c_{i,j}$ not all equal to zero, such that

$$\sum_i \sum_j c_{i,j} \phi_i \xi_{i,j} = 0. \tag{15}$$

Recall the ball $\omega_i^\rho \subset \omega_i$, on which $\phi_i = 1$. Thus, on some ω_i^ρ, $\sum_j c_{i,j}\xi_{i,j} = 0$ for $c_{i,j}$ not all equal to zero, which indicates that the polynomials $\xi_{i,j}$, $1 \leqslant j \leqslant m_i$, are linearly dependent. This contradicts the fact that $\xi_{i,j}$, $1 \leqslant j \leqslant m_i$, are basis functions of the space V_i. □

Recall the GFEM satisfying the above four assumptions yields the linear system (9). Many iterative methods, such as the conjugate gradient method, multigrid methods (e.g., [15]), may be used to solve this linear system of equations, while the convergence property of these iterative methods is often related to the condition number of the stiffness matrix. From now on, we will concentrate on the estimates of the condition number.

Note that by Eq. (10), the coercivity of $a(\cdot,\cdot)$ in (3), and Proposition 2.7, it is clear that $\tilde{\mathcal{A}}$ is a real symmetric positive definite matrix. Therefore, all its eigenvalues are positive. Let

$$\lambda_{max} = \max_{\|\mathbf{x}\|_{l^2}=1}\left(\mathbf{x}^T\tilde{\mathcal{A}}\mathbf{x}\right), \qquad \lambda_{min} = \min_{\|\mathbf{x}\|_{l^2}=1}\left(\mathbf{x}^T\tilde{\mathcal{A}}\mathbf{x}\right) \tag{16}$$

be the largest and the smallest eigenvalues, respectively. Then, the l^2-condition number $\kappa(\tilde{\mathcal{A}})$ of the stiffness matrix is given by

$$\kappa(\tilde{\mathcal{A}}) = \lambda_{max}/\lambda_{min}. \tag{17}$$

With the notation and assumptions presented above, we shall estimate the conditioning of the linear system (9) in the next section.

3. Estimates of the condition numbers

In this section, motivated by [2,1,7,21], we provide upper bounds (Theorem 3.10) on the condition number of the linear system in the GFEMs satisfying the assumptions in Section 2.

3.1. Lemmas

We shall first present some existing embedding results in different Sobolev spaces (Lemma 3.1) and then derive critical estimates specific to the GFEMs under consideration (Lemmas 3.3 and 3.4) that are needed to carry out further analysis.

Recall that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain. The following result is a special case of the Sobolev embedding theorems (see [14,29] for the proof).

Lemma 3.1. *For any $v \in W^{1,p}(\Omega)$, there are constants $\beta_1, \beta_2 > 0$, depending on n and Ω, but not on v, such that*

$$\|v\|_{L^{np/(n-p)}(\Omega)} \leqslant \beta_1(n-p)^{-1}\|v\|_{W^{1,p}(\Omega)}, \quad n > p,$$
$$\|v\|_{L^q(\Omega)} \leqslant \beta_2 q^{1-1/n}\|v\|_{W^{1,p}(\Omega)}, \quad p = n,\ q \geqslant 1.$$

Let $\omega_i \subset \Omega$ be a patch defined in Definition 2.1, satisfying Assumptions 1–4. Recall its reference patch $\hat{\omega}_i := \{\hat{x} = (x - x_i^\rho)/h_i,\ \forall x \in \omega_i\}$ and the function $\hat{v}(\hat{x}) := v(x)$, where x_i^ρ is the center of the ball $\omega_i^\rho \subset \omega_i$ from Assumption 1. We now have the following scaling argument.

Lemma 3.2. *For any $v \in W^{k,p}(\omega_i)$, $1 \leqslant p \leqslant \infty$,*

$$|\hat{v}|_{W^{k,p}(\hat{\omega}_i)} = h_i^{k-(n/p)}|v|_{W^{k,p}(\omega_i)}.$$

Proof. The proof is standard. For $p < \infty$,

$$\begin{aligned}
|\hat{v}|^p_{W^{k,p}(\hat{\omega}_i)} &= \sum_{|\alpha|=k}\int_{\hat{\omega}_i}\left(\partial^\alpha\hat{v}(\hat{x})\right)^p d\hat{x} = \sum_{|\alpha|=k}\int_{\hat{\omega}_i}\left(\partial^\alpha v\left(h_i\hat{x} + x_i^\rho\right)\right)^p d\hat{x} \\
&= \sum_{|\alpha|=k}\int_{\omega_i} h_i^{kp}\left(\partial^\alpha v(x)\right)^p h_i^{-n}\,dx = h_i^{kp-n}\sum_{|\alpha|=k}\int_{\omega_i}\left(\partial^\alpha v(x)\right)^p dx \\
&= h_i^{kp-n}|v|^p_{W^{k,p}(\omega_i)}.
\end{aligned}$$

For $p = \infty$,

$$|\hat{v}|_{W^{k,\infty}(\hat{\omega}_i)} = \sup_{|\alpha|=k} \left|\partial^\alpha \hat{v}(\hat{x})\right| = \sup_{|\alpha|=k} \left|\partial^\alpha v\left(h_i \hat{x} + x_i^\rho\right)\right|$$
$$= h_i^k \sup_{|\alpha|=k} \left|\partial^\alpha v(x)\right| = h_i^k |v|_{W^{k,\infty}(\omega_i)}. \qquad \square$$

Recall the GFE space S_G from (7) and the dimension m_i of the local approximation space V_i on the patch ω_i in Definition 2.1. Note that any $v \in S_G$ can be written as

$$v = \sum_i \sum_{j \leqslant m_i} \tilde{c}_{i,j} \phi_i \xi_{i,j},$$

where ϕ_i and $\xi_{i,j}$ are defined in Assumption 1 and (11), respectively, and $\tilde{c}_{i,j} \in \mathbb{R}$. Define

$$\mathcal{N}(i) := \{j,\ \omega_j \cap \omega_i \neq \emptyset\} \tag{18}$$

to be the set of indices of the neighbor patches of ω_i.

Recall the reference patch $\hat{\omega}_i$. Define the space on $\hat{\omega}_i$

$$\hat{Q}_i(\hat{\omega}_i) := \operatorname{span}\left\{\hat{\phi}_k \hat{\xi}_{k,j}|_{\hat{\omega}_i},\ k \in \mathcal{N}(i),\ 1 \leqslant j \leqslant m_k\right\}.$$

It is clear that $\hat{Q}_i(\hat{\omega}_i)$ is a finite-dimensional space. We say $\hat{Q}_l(\hat{\omega}_l)$ is a generator of $\hat{Q}_i(\hat{\omega}_i)$ if $\{\hat{\omega}_l, \hat{S}_l\}$ and $\{\hat{\omega}_i, \hat{S}_i\}$ belong to the same class (see Assumption 4). We then have the following inverse estimates.

Lemma 3.3. *For any $v \in S_G$, there exist constants $\beta_3, \beta_4 > 0$, bounded for all $1 \leqslant i \leqslant I$, such that for $q \geqslant 1$,*

$$\|v\|_{H^1(\omega_i)} \leqslant \beta_3 h_i^{-1+n/2} \|v\|_{L^\infty(\omega_i)} \leqslant \beta_4 h_i^{-1+n/2-n/q} \|v\|_{L^q(\omega_i)},$$
$$\|v\|_{H^1(\omega_i)} \leqslant \beta_3 h_i^{-1+n/2} \|v\|_{L^\infty(\omega_i)} \leqslant \beta_4 \|v\|_{L^{2n/(n-2)}(\omega_i)}, \quad n \geqslant 3.$$

Proof. This is a generalization of the inverse inequalities in the usual finite element analysis. We here want to show that the constants in the above inequalities are uniformly bounded for all $1 \leqslant i \leqslant I$.

By Assumption 4, there is a finite selection $\{\hat{Q}_l(\hat{\omega}_l)\}$ of generators for the set $\{\hat{Q}_i(\hat{\omega}_i),\ 1 \leqslant i \leqslant I\}$. The cardinality of the generator set $\{\hat{Q}_l(\hat{\omega}_l)\}$ is independent of the dimension of the GFE space S_G. Therefore, it suffices to show that these constants are uniformly bounded for those spaces $\hat{Q}_i(\hat{\omega}_i)$ that share the same generator $\hat{Q}_l(\hat{\omega}_l)$.

Suppose $\hat{Q}_l(\hat{\omega}_l)$ is the generator of $\hat{Q}_i(\hat{\omega}_i)$. By the equivalence of the norms on the finite-dimensional spaces $\hat{Q}_l(\hat{\omega}_l)$, there exist constants $M_1, M_2 > 0$, such that for any $\hat{v} \in \hat{Q}_l(\hat{\omega}_l)$,

$$\|\hat{v}\|_{H^1(\hat{\omega}_l)} \leqslant M_1 \|\hat{v}\|_{L^\infty(\hat{\omega}_l)} \leqslant M_2 \|\hat{v}\|_{L^q(\hat{\omega}_l)}, \tag{19}$$

where M_1 and M_2 depend on $\hat{Q}_l(\hat{\omega}_l)$ and q.

Then, there exists a rotation matrix $\mathbf{R}$, such that for any $\hat{x}_i \in \hat{\omega}_i$ and $\hat{v}_i \in \hat{Q}_i(\hat{\omega}_i)$,

$$\hat{x}_l := \mathbf{R}\hat{x}_i \in \hat{\omega}_l \quad \text{and} \quad \hat{Q}_l(\hat{\omega}_l) \ni \hat{v}_l(\hat{x}_l) := \hat{v}_i(\hat{x}_i).$$

Therefore, by the usual estimates on Sobolev semi-norms (Theorem 3.1.2 [9]), there exist constants M_3 and M_4, depending on m and n, but not on $\mathbf{R}$ such that for $1 \leqslant p \leqslant \infty$,

$$M_3 \|\mathbf{R}\|^{-m} |\det \mathbf{R}|^{1/p} |\hat{v}_i|_{W^{m,p}(\hat{\omega}_i)} \leqslant |\hat{v}_l|_{W^{m,p}(\hat{\omega}_l)} \leqslant M_4 \left\|\mathbf{R}^{-1}\right\|^m |\det \mathbf{R}|^{1/p} |\hat{v}_i|_{W^{m,p}(\hat{\omega}_i)}.$$

Since $\mathbf{R}$ is an $n \times n$ rotation matrix, the above inequalities read

$$M_3 |\hat{v}_i|_{W^{m,p}(\hat{\omega}_i)} \leqslant |\hat{v}_l|_{W^{m,p}(\hat{\omega}_l)} \leqslant M_4 |\hat{v}_i|_{W^{m,p}(\hat{\omega}_i)}. \tag{20}$$

Then, using (19) and (20), we have

$$\|\hat{v}_i\|_{H^1(\hat{\omega}_i)} \leqslant M_5 \|\hat{v}_i\|_{L^\infty(\hat{\omega}_i)} \leqslant M_6 \|\hat{v}_i\|_{L^q(\hat{\omega}_i)},$$

where M_5 and M_6 depend on $\hat{Q}_l(\hat{\omega}_l)$, n, m, and q, but not on i.

Therefore, for $h_i < 1$, using Lemma 3.2, we have positive constants β_3 and β_4, for all $1 \leqslant i \leqslant I$, such that for any $v \in S_G$,

$$\|v\|_{H^1(\omega_i)} \leqslant \beta_3 h_i^{-1+n/2} \|v\|_{L^\infty(\omega_i)} \leqslant \beta_4 h_i^{-1+n/2-n/q} \|v\|_{L^q(\omega_i)}.$$

In the case of $n \geqslant 3$, we let $q = 2n/(n-2)$ in the above inequality. Then, for any $v \in S_G$,

$$\|v\|_{H^1(\omega_i)} \leqslant \beta_3 h_i^{-1+n/2} \|v\|_{L^\infty(\omega_i)} \leqslant \beta_4 \|v\|_{L^{2n/(n-2)}(\omega_i)},$$

which completes the proof. $\square$

Recall the partition of unity $\{\phi_i\}$ from Assumption 2. To better present the result, we introduce the scaled function $\varphi_i(x) := h_i^{(2-n)/2}\phi_i(x)$ on ω_i. Then, for any $v \in S_G$, we can write

$$v = \sum_i \sum_j \tilde{c}_{i,j}\phi_i\xi_{i,j} = \sum_i \sum_j c_{i,j}\varphi_i\xi_{i,j}, \tag{21}$$

where $c_{i,j} = h_i^{(n-2)/2}\tilde{c}_{i,j}$. Then, we have the following critical estimates for the scaled coefficients $c_{i,j}$.

Lemma 3.4. *There exist $\beta_5, \beta_6 > 0$, for all $1 \leqslant i \leqslant I$, such that for any $v \in S_G$, we have*

$$\beta_5 \sum_{j\leqslant m_i} c_{i,j}^2 \leqslant h_i^{n-2}\|v\|^2_{L^\infty(\omega_i)} \leqslant \beta_6 \sum_{k\in\mathcal{N}(i)} \sum_{j\leqslant m_k} c_{k,j}^2.$$

Proof. We first prove the second inequality. Note that on ω_i,

$$v = \sum_{k\in\mathcal{N}(i)} \sum_{j\leqslant m_k} c_{k,j}\varphi_k\xi_{k,j}.$$

By Assumption 4, the size of neighbor patches of ω_i is comparable with h_i. Then, From (6) in Definition 2.1 and (11), we have $|\varphi_k\xi_{k,j}| \leqslant M_1 h_i^{(2-n)/2}$ for any $k \in \mathcal{N}(i)$, where M_1 is bounded for all $1 \leqslant i \leqslant I$. Therefore,

$$\begin{aligned}\|v\|^2_{L^\infty(\omega_i)} &= \Bigg\| \sum_{k\in\mathcal{N}(i)} \sum_{j\leqslant m_k} c_{k,j}\varphi_k\xi_{k,j} \Bigg\|^2_{L^\infty(\omega_i)} \leqslant M_1 h_i^{2-n} \Bigg(\sum_{k\in\mathcal{N}(i)} \sum_{j\leqslant m_k} |c_{k,j}| \Bigg)^2 \\ &\leqslant \beta_6 h_i^{2-n} \sum_{k\in\mathcal{N}(i)} \sum_{j\leqslant m_k} c_{k,j}^2,\end{aligned}$$

where β_6 depends on M_1 and the number of the shape functions whose supports intersect ω_i. Since by Assumption 4, the number of classes of the data sets $\{\omega_i, S_i\}$ is independent of the dimension of the GFE space, β_6 can be chosen to be bounded for all $1 \leqslant i \leqslant I$.

We now prove the first inequality by showing that β_5 is bounded away from 0. Recall the reference patch $\hat{\omega}_i$, the function $\hat{v}(\hat{x})$, and the ball ω_i^ρ from Assumption 1. Thus, for any $v = \sum_i \sum_j \tilde{c}_{i,j}\phi_i\xi_{i,j} \in S_G$, we have

$$\hat{v}(\hat{x}) = \sum_{k\in\mathcal{N}(i)} \sum_j \tilde{c}_{k,j}(\hat{\phi}_k\hat{\xi}_{k,j})(\hat{x}), \quad \forall \hat{x} \in \hat{\omega}_i.$$

We first show the existence of the lower bound of $\beta_5 > 0$ on $\hat{\omega}_i$, such that

$$\beta_5 \sum_{j\leqslant m_i} \tilde{c}_{i,j}^2 \leqslant \|\hat{v}\|^2_{L^\infty(\hat{\omega}_i)}. \tag{22}$$

Assume that β_5 has no lower bound. Then, there is a sequence $(\tilde{c}_{i,j}^l)$, $\sum_{j\leqslant m_i}(\tilde{c}_{i,j}^l)^2 = 1$, $l \to \infty$, such that for l large

$$\|\hat{v}_l\|^2_{L^\infty(\hat{\omega}_i^\rho)} \leqslant \|\hat{v}_l\|^2_{L^\infty(\hat{\omega}_i)} \leqslant 1/l, \tag{23}$$

where $\hat{v}_l = \sum_{k\in\mathcal{N}(i)} \sum_j \tilde{c}_{k,j}^l \hat{\phi}_k\hat{\xi}_{k,j}$ on $\hat{\omega}_i$. Note that $(\tilde{c}_{i,j}^l) \in \mathbb{R}^{m_i}$ is a bounded sequence (with index l) in a finite-dimensional space. Therefore, there exists a subsequence $(\tilde{c}_{i,j}^{l'})$ converging to $(c_{i,j}^F)$, where $c_{i,j}^F = \lim_{l'\to\infty} c_{i,j}^{l'}$, for all $j \leqslant m_i$.

On the ball $\hat{\omega}_i^\rho$, let

$$\hat{v}^F := \sum_{j\leqslant m_i} c_{i,j}^F \hat{\phi}_i\hat{\xi}_{i,j}.$$

Thus, by (23), $\|\hat{v}^F\|_{L^\infty(\hat{\omega}_i^\rho)} = 0$. Since $\hat{v}^F$ is a polynomial on $\hat{\omega}_i^\rho$ and the basis functions $\hat{\xi}_{i,j}$ are linear independent, we conclude $c_{i,j}^F = 0$. This is a contradiction that $\sum_{j\leqslant m_i}(c_{i,j}^F)^2 = 1$, which completes the proof for the existence of the lower bound of β_5 in (22).

Since $\|\hat{v}\|_{L^\infty(\hat{\omega}_i)} = \|v\|_{L^\infty(\omega_i)}$, we have proved that there is a lower bound for $\beta_5 > 0$ on ω_i, such that

$$\beta_5 \sum_{j\leqslant m_i} c_{i,j}^2 = \beta_5 h_i^{n-2} \sum_{j\leqslant m_i} (\tilde{c}_{i,j})^2 \leqslant h_i^{n-2}\|v\|^2_{L^\infty(\omega_i)}. \tag{24}$$

It can be seen that the lower bound of β_5 depends on $\hat{\omega}_i$ and the polynomial space span$\{\hat{\xi}_{i,j}\}$ defined on it. Since the number of classes of the data sets $\{\omega_i, S_i\}$ is independent of the dimension of the GFE space, we can choose a global lower bound for $\beta_5 > 0$, such that (24) holds for each i. This completes the proof of this lemma. □

3.2. Condition numbers

Recall that we are interested in the conditioning of the linear system from the GFEM. To better present our result, we shall estimate the condition numbers of the following stiffness matrix $\mathcal{A}$ and the mass matrix $\mathcal{M}$,

$$\mathcal{A} = \big(a(\varphi_i\xi_{i,j}, \varphi_l\xi_{l,k})\big), \qquad \mathcal{M} = \big((\varphi_i\xi_{i,j}, \varphi_l\xi_{l,k})_{L^2}\big). \tag{25}$$

Theorem 3.5. *The condition number of the mass matrix,*

$$\kappa(\mathcal{M}) \leqslant Ch_{max}^2/h_{min}^2,$$

where h_{max} and h_{min} are the largest and the smallest diameters among all patches.

Proof. Recall for any $v \in S_G$, $v = \sum_i \sum_{j\leqslant m_i} c_{i,j}\varphi_i\xi_{i,j}$, where $\varphi_i = h_i^{(2-n)/2}\phi_i$ is a scaled function. Let $\mathbf{V} := (c_{i,j})$ be the vector containing all the coefficients $c_{i,j}$. Then

$$\begin{aligned}\mathbf{V}^T\mathcal{M}\mathbf{V} &= \|v\|^2_{L^2(\Omega)} \leqslant M_1\sum_i \|v\|^2_{L^2(\omega_i)} \leqslant M_2\sum_i h_i^2\|v\|^2_{H^1(\omega_i)}\\ &\leqslant M_3\sum_i h_i^n\|v\|^2_{L^\infty(\omega_i)} \leqslant M_4\sum_i\sum_{k\in\mathcal{N}(i)}\sum_{j\leqslant m_k} h_i^2 c_{k,j}^2\\ &\leqslant M_5 h_{max}^2\sum_i\sum_j c_{i,j}^2 = Ch_{max}^2\mathbf{V}^T\mathbf{V}.\end{aligned}$$

The first inequality is based on Definition 2.1; the second inequality is due to the scaling argument for functions in L^2 and H^1 in Lemma 3.2; the third inequality is based on Lemma 3.3; the fourth inequality is from Lemma 3.4. Therefore, we have

$$\lambda_{max}(\mathcal{M}) \leqslant Ch_{max}^2,$$

where $\lambda_{max}(\mathcal{M})$ is the largest eigenvalue defined in (16).

For the smallest eigenvalue, we have

$$\begin{aligned}\mathbf{V}^T\mathbf{V} &= \sum_i\sum_j c_{i,j}^2 \leqslant M_1\sum_i h_i^{n-2}\|v\|^2_{L^\infty(\omega_i)}\\ &\leqslant M_2\sum_i h_i^{n-2}h_i^{-n}\|v\|^2_{L^2(\omega_i)} = M_2\sum_i h_i^{-2}\|v\|^2_{L^2(\omega_i)}\\ &\leqslant M_2 h_{min}^{-2}\sum_i\|v\|^2_{L^2(\omega_i)} \leqslant Ch_{min}^{-2}\|v\|^2_{L^2(\Omega)} = Ch_{min}^{-2}\mathbf{V}^T\mathcal{M}\mathbf{V}.\end{aligned}$$

The second inequality is due to Lemma 3.4; the third inequality is based on the inverse inequality from Lemma 3.3. Consequently,

$$\lambda_{min}(\mathcal{M}) \geqslant Ch_{min}^2,$$

which completes the proof. □

Therefore, the condition number of the mass matrix $\mathcal{M}$ of the GFEM asymptotically depend on the ratio of the largest to the smallest diameter in the open cover set $\{\omega_i\}$. We now proceed to the estimate for the stiffness matrix $\mathcal{A}$. In the text below, we denote by $N := \dim(S_G)$ the dimension of the GFE space S_G.

Lemma 3.6. *For the stiffness matrix $\mathcal{A}$ defined in* (25)*,*

$$\lambda_{max}(\mathcal{A}) = \mathcal{O}(1).$$

Proof. For any $v \in S_G$, $v = \sum_i \sum_{j\leqslant m_i} c_{i,j}\varphi_i\xi_{i,j}$. Recall $\mathbf{V} := (c_{i,j})$. Then,

$$\begin{aligned}\mathbf{V}^T\mathcal{A}\mathbf{V} &= a(v,v) \leqslant M_1\|v\|^2_{H^1(\Omega)} \leqslant M_1\sum_i\|v\|^2_{H^1(\omega_i)}\\ &\leqslant M_2\sum_i h_i^{n-2}\|v\|^2_{L^\infty(\omega_i)} \leqslant M_3\sum_i\sum_{k\in\mathcal{N}(i)}\sum_j c_{k,j}^2 \leqslant C\mathbf{V}^T\mathbf{V}.\end{aligned}$$

The above proof is based on the continuity of the bilinear form $a(\cdot,\cdot)$, the inverse inequality in Lemma 3.3, and the estimate in Lemma 3.4. □

Lemma 3.6 holds for any n, while for the investigation of the smallest eigenvalue of $\mathcal{A}$, we shall have to consider for different values of n, due to different Sobolev inequalities associated to each case.

Lemma 3.7. *For $n = 1$, the smallest eigenvalue of the stiffness matrix $\mathcal{A}$,*

$$\lambda_{min}(\mathcal{A}) \geqslant CN^{-1}h_{min},$$

where $C > 0$ depends on Ω and the GFE space S_G.

Proof. For any $v \in S_G$, $v = \sum_i \sum_{j \leqslant m_i} c_{i,j}\varphi_i\xi_{i,j}$, $\mathbf{V} := (c_{i,j})$, we have

$$\begin{aligned}\mathbf{V}^T\mathbf{V} &= \sum_i \sum_j c_{i,j}^2 \leqslant M_1 \sum_i h_i^{-1}\|v\|_{L^\infty(\omega_i)}^2 \\ &\leqslant M_1 h_{min}^{-1}\|v\|_{L^\infty(\Omega)}^2 \sum_i 1 \leqslant M_2 h_{min}^{-1} N\|v\|_{H^1(\Omega)}^2 \\ &\leqslant M_3 h_{min}^{-1} N a(v,v) \leqslant C h_{min}^{-1} N \mathbf{V}^T\mathcal{A}\mathbf{V}.\end{aligned}$$

Lemma 3.4, the coercivity of the bilinear form $a(\cdot,\cdot)$, and the Sobolev embedding theorem are applied in the proof. It is evident that we can choose $M_1 = \beta_5^{-1}$ from Lemma 3.4, which depends on the choice of S_G. Therefore, the constant C is determined by S_G, the Sobolev embedding constant on Ω, and the connection between the form $a(\cdot,\cdot)$ and the H^1 norm. □

Lemma 3.8. *For $n = 2$, the smallest eigenvalue of the stiffness matrix $\mathcal{A}$,*

$$\lambda_{min}(\mathcal{A}) \geqslant CN^{-1}\bigl(1 + \bigl|\log\bigl(Nh_{min}^2\bigr)\bigr|\bigr)^{-1}.$$

Proof. Let $2 < q < \infty$. For any $v \in S_G$, $v = \sum_i \sum_{j \leqslant m_i} c_{i,j}\varphi_i\xi_{i,j}$, $\mathbf{V} := (c_{i,j})$, we have

$$\begin{aligned}\mathbf{V}^T\mathbf{V} &= \sum_i \sum_j c_{i,j}^2 \leqslant M_1 \sum_i \|v\|_{L^\infty(\omega_i)}^2 \leqslant M_2 \sum_i h_i^{-4/q}\|v\|_{L^q(\omega_i)}^2 \\ &\leqslant M_3 \left(\sum_i \bigl(h_i^{-4/q}\bigr)^{q/(q-2)}\right)^{(q-2)/q} \left(\sum_i \|v\|_{L^q(\omega_i)}^q\right)^{2/q} \\ &\leqslant M_4 \left(\sum_i h_i^{-4/(q-2)}\right)^{(q-2)/q} \|v\|_{L^q(\Omega)}^2 \\ &\leqslant M_5 \left(\sum_i h_i^{-4/(q-2)}\right)^{(q-2)/q} q\|v\|_{H^1(\Omega)}^2 \\ &\leqslant M_6 h_{min}^{-4/q} N^{(q-2)/q} q\mathbf{V}^T\mathcal{A}\mathbf{V} = M_6 N\bigl(Nh_{min}^2\bigr)^{-2/q} q\mathbf{V}^T\mathcal{A}\mathbf{V}.\end{aligned}$$

The above estimates are based on the inverse estimate in Lemma 3.3, Hölder's inequality, and the Sobolev embedding estimate from Lemma 3.1. Now, for any $\epsilon > 0$, if we choose $q = \max(2+\epsilon, |\log(Nh_{min}^2)|)$, the desired result follows. □

Lemma 3.9. *For $n \geqslant 3$, the smallest eigenvalue of the stiffness matrix $\mathcal{A}$,*

$$\lambda_{min}(\mathcal{A}) \geqslant CN^{-2/n},$$

where $C > 0$ depends on Ω and the GFE space S_G.

Proof. Let $2 < q < \infty$. Similarly, for any $v \in S_G$, $v = \sum_i \sum_{j \leqslant m_i} c_{i,j}\varphi_i\xi_{i,j}$, $\mathbf{V} := (c_{i,j})$, we have

$$\begin{aligned}\mathbf{V}^T\mathbf{V} &= \sum_i \sum_j c_{i,j}^2 \leqslant M_1 \sum_i h_i^{n-2}\|v\|_{L^\infty(\omega_i)}^2 \leqslant M_2 \sum_i \|v\|_{L^{\frac{2n}{n-2}}(\omega_i)}^2 \\ &\leqslant M_2 \left(\sum_i 1\right)^{\frac{2}{n}} \left(\sum_i \|v\|_{L^{\frac{2n}{n-2}}(\omega_i)}^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leqslant M_3 N^{\frac{2}{n}}\|v\|_{L^{\frac{2n}{n-2}}(\Omega)}^2 \\ &\leqslant M_4 N^{\frac{2}{n}}\|v\|_{H^1(\Omega)}^2 \leqslant CN^{\frac{2}{n}}\mathbf{V}^T\mathcal{A}\mathbf{V}.\end{aligned}$$

The proof is carried out by Lemma 3.4, the inverse estimate in Lemma 3.3, Hölder's inequality, and the Sobolev embedding result from Lemma 3.1. □

With the above estimates on the largest and smallest eigenvalues for different n, we then obtain the main estimate on the upper bounds of the condition number of the stiffness matrix in a class of GFEMs.

Theorem 3.10. *Under Assumptions* 1–4 *on the GFEM solving Eq.* (1)*, let*

$$\mathcal{A} = \big(a(\varphi_i \xi_{i,j}, \varphi_l \xi_{l,k})\big)$$

be the scaled stiffness matrix and $N := \dim(S_G)$*. Suppose the bilinear form* $a(\cdot,\cdot)$ *is continuous and coercive on* $H^1(\Omega)$*. Then the condition number* $\kappa(\mathcal{A})$ *satisfies*

$$\kappa(\mathcal{A}) \leqslant C N h_{min}^{-1}, \quad n = 1;$$
$$\kappa(\mathcal{A}) \leqslant C N\big(1 + \big|\log\big(N h_{min}^2\big)\big|\big), \quad n = 2;$$
$$\kappa(\mathcal{A}) \leqslant C N^{2/n}, \quad n \geqslant 3.$$

The constant C *depends on the specific selection of the partition of unity, patches, and local approximation spaces, but not on* N *or* h_{min}*.*

Proof. Using $\kappa(\mathcal{A}) = \lambda_{max}(\mathcal{A})/\lambda_{min}(\mathcal{A})$, we obtain these estimates by direct calculations based on the results from the above lemmas. □

Remark 3.11. It can be seen that the asymptotic behavior of the condition number of the scaled stiffness matrix $\mathcal{A}$ in the GFEMs satisfying Assumption 1–4 is similar to that in the usual FEMs [7]. One can further show that the above estimates are sharp by constructing specific functions in the GFE space on quasi-uniform patches. It can be done by considering a function with very high frequency (e.g., set a particular $c_{i,j} = 1$ and other coefficients 0) and a function with low frequency (e.g., construct a function with the Dirichlet boundary condition such that the function is a constant on all patches away from the boundary).

Remark 3.12. Different from the usual FEM, the constant C in the estimates for the GFEM may vary a lot, because of different selections of the patch ω_i, the partition of unity function ϕ_i, and the local approximation space. For example, if the constant C_1 in (6) is large, the maximum eigenvalue of $\mathcal{A}$ (Lemma 3.6) will be large in magnitude, which in turn leads to bad conditioning of the linear system. This has been observed in practical computations when the over-lapping regions between the intersecting patches are small or when certain boundary conditions have to be imposed. In this case, the asymptotic growing factor in the above estimates may not be the dominating contributor to the magnitude of the condition number. Note that some of our techniques may be extended for the investigation of the effect of different parameters in the GFEM on the condition number. For example, it is possible to obtain sharper estimates on the magnitude of the condition number in relation to the bounds of the derivatives of the partition of unity function for some specific GFEMs, which we will report in a forthcoming paper. In Section 4, the numerical tests actually show some interesting phenomena in this aspect.

4. Numerical illustrations

We present numerical results that illustrate our estimates on the condition numbers in Theorems 3.5 and 3.10. These tests confirm our theory and also show interesting relations on the choice of the partition of unity function and the magnitude of the corresponding condition numbers.

4.1. Numerical tests

We performed the GFEM solving Poisson's equation with the zero Dirichlet boundary condition on a 1-D domain $\Omega_1 = (0,1)$ and a 2-D domain $\Omega_2 = (0,1) \times (0,1)$ using quasi-uniform patches. This enables us to simplify the test and still obtain relevant results.

For the 1-D problem, let $I \geqslant 3$ be the number of the quasi-uniform patches. Let $h = 1/(I-1)$. Then, we choose the patches $\omega_i = ((i-2)h, ih)$, $2 \leqslant i \leqslant I-1$, $\omega_1 = (0,h)$, and $\omega_I = (1-h,1)$ (see Fig. 1 for example). The partition of unity function ϕ_i on each ω_i is a translation and dilation of the following reference function defined on $(-1,1)$,

$$\hat{\phi}(x) = \begin{cases} 1 & -r \leqslant x \leqslant r, \\ \big(1 - \big(\frac{x-r}{1-2r}\big)^k\big)^k & r < x < 1-r, \\ 1 - \big(1 - \big(\frac{x-r+1}{1-2r}\big)^k\big)^k & -1+r < x < -r, \\ 0 & \text{otherwise}, \end{cases} \tag{26}$$

Table 1
Condition numbers of the stiffness matrix $\mathcal{A}$ in the 1-D GFEM.

$1/h$	Growth history for $k=1$			Growth history for $k=2$		
	$r=0.1$	$r=0.3$	$r=0.4$	$r=0.1$	$r=0.3$	$r=0.4$
5	62	128	259	79	152	268
10	235	470	945	272	545	1078
20	853	1706	3419	1025	2077	4132
40	3323	6647	13 296	4058	8167	16 210
80	13 131	26 263	52 527	16 035	32 028	64 039

Table 2
Condition numbers of the stiffness matrix $\mathcal{A}$ and the mass matrix $\mathcal{M}$ in the 2-D GFEM.

$1/h$	Growth history for $\mathcal{A}$, $k=4$			Growth history for $\mathcal{M}$, $k=1$		
	$r=0.1$	$r=0.3$	$r=0.4$	$r=0.1$	$r=0.3$	$r=0.4$
10	3778	8555	17 265	1569	367	275
20	16 410	37 453	76 409	1530	370	277
40	72 412	166 031	341 372	1534	371	277
80	303 913	673 926	1 440 632	1535	371	277

where $0 < r < 0.5$ is the radius of the ball $\hat{\omega}^{\rho}$, on which $\hat{\phi} = 1$. Then, each ϕ_i in the GFEM is defined as $\phi_i(x) = \hat{\phi}([x - (i-1)h]/h)$, for any $x \in \omega_i$. The parameter k in (26) is to determine the regularity of the function. Hence, it is clear that the set $\{\phi_i\}$ forms a flat-top partition of unity of Ω_1 and it resembles the partition of unity in Fig. 1. For the local approximation space V_i, we used the space of linear functions with the basis functions $\xi_{i,1} = 1$ and $\xi_{i,2} = (x - (i-1)h)/h$ on ω_i, for $2 \leqslant i \leqslant I-1$; on the two patches ω_1 and ω_I near the boundary, the corresponding spaces are $V_1 = \mathrm{span}\{x/h\}$ and $V_I = \mathrm{span}\{(x-1)/h\}$, in order to impose the Dirichlet boundary condition.

The condition numbers of the stiffness matrix $\mathcal{A}$ in the proposed 1-D GFEMs are listed in Table 1, for different patch sizes h and different shapes of partition of unity functions. In each column of the table, for a fixed r and k, it is clear that the condition numbers are increasing by a factor of 4, as the number of patches increases by a factor of 2. This perfectly matches the estimates in Theorem 3.10, since the patch size decreases by a factor of 2 while the dimension of the GFE space increases by a factor of 2 at the same time.

For the 2-D problem with $\Omega_2 = (0,1) \times (0,1)$, we consider the patches of the form $\omega_{i,j} = \omega_i \times \omega_j$, $1 \leqslant i \leqslant I$, $1 \leqslant j \leqslant I$, where ω_i and ω_j are the patches defined for the above 1-D problem on Ω_1. Therefore, the patches $\omega_{i,j}$ on Ω_2 are rectangles (most of them are squares) with the length of their largest side $= 2h$ for $h = 1/(I-1)$. Similarly, we define the associated partition of unity function $\phi_{i,j}(x,y) = \phi_i(x)\phi_j(y)$, where ϕ_i and ϕ_j are partition of unity functions in the 1-D problem associated to ω_i and ω_j, respectively. The fact that $\{\phi_{i,j}\}$ is a partition of unity can be verified by summing up all $\phi_{i,j}$ over the domain Ω_2. For the local approximation space associated to $\omega_{i,j}$, we used the linear function space $V_{i,j} = \mathrm{span}\{1, x, y\}$ with basis functions in the format as in (11) and also imposed the zero Dirichlet condition on the boundary of the unit square Ω_2.

The numerical results in 2-D also verify our estimates on the asymptotic behavior of the condition numbers. In Table 2, it is clear that for a fixed r and k, the condition numbers of the stiffness matrix $\mathcal{A}$ grow by a factor of 4. Note that the dimension N of the GFE space $= \mathcal{O}(h^{-2})$ in our model problem. Thus, this is in fact the growth rate given in Theorem 3.10 for the 2-D case. We also tested for the mass matrix $\mathcal{M}$ for the 2-D problem. Based on Theorem 3.5, the condition numbers of $\mathcal{M}$, for a fixed r and k, should not vary much for different values of h, since the patches are quasi-uniform. This prediction is confirmed in Table 2. Namely, the condition number of the mass matrix $= \mathcal{O}(1)$, for a fixed r, independent of the dimension of the approximation space, for the model problem.

In addition to verifying our theory, we also observe that the gradient of the partition of unity function (different values of r) dramatically affects the magnitude of the condition number of the stiffness matrix in an interesting pattern. Recall the constant C_1 in (6) for the gradient of the function ϕ_i. Tables 1 and 2 show that the larger the constant C_1 is (larger r), the larger the magnitude of the condition numbers of $\mathcal{A}$. A more careful study using these analytical tools is expected to provide a better understanding on the relation between the condition number and other parameters in the GFEM.

Remark 4.1. Assumptions 1–4 in Section 2 cover a class of GFEMs that can be quite general and complex. Our focus in this section, rather than the complexity of these GFEMs, is to show numerically the asymptotic behavior of the condition numbers predicted by the theorems in Section 3. Therefore, we chose relatively simple GFEMs, in which we can demonstrate the theory with a lower computation cost. Note that the data sets $\{\omega_i, S_i\}_{i=2}^{I-1}$ in the 1-D example or $\{\omega_{i,j}, S_{i,j}\}_{i,j=2}^{I-1}$ in the 2-D example belong to the same class, since they are generated by same shape functions on the same reference domain by translation and dilation. Moreover, the number of the classes does not grow when we increase the dimension of the GFE space (i.e., decrease h as in Tables 1 and 2). For more examples of the GFEMs satisfying these assumptions, see [2,24] and references therein.

4.2. Conclusion

We investigated the asymptotic behavior of the condition numbers for the stiffness matrix and the mass matrix in a class of GFEMs that are widely used in practical computations, in relation to the dimension of the approximation space. Utilizing different parameters in the partition of unity function (e.g., different shapes and different smoothness), we implemented the corresponding GFEMs solving Poisson's equation with the Dirichlet boundary condition in different space dimensions. All the numerical results show that the theoretical rate of growth for the condition numbers is consistent with our calculations. In addition, these tables show the strong dependence of the condition number on the choice of the partition of unity (e.g., different shapes). A further study in this direction will certainly be of great importance in improving the GFEMs for practical use.

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References

[1] I. Babuška, U. Banerjee, J. Osborn, Survey of meshless and generalized finite element methods: a unified approach, Acta Numer. 12 (2003) 1–125.
[2] I. Babuška, U. Banerjee, J.E. Osborn, Generalized finite element methods: main ideas, results, and perspective, ICES Reports, 2004.
[3] I. Babuška, U. Banerjee, J.E. Osborn, Superconvergence in the generalized finite element method, Numer. Math. 107 (2007) 353–395.
[4] I. Babuška, G. Caloz, J.E. Osborn, Special finite element methods for a class of second order elliptic problems with rough coefficients, SIAM J. Numer. Anal. 31 (1994) 945–981.
[5] I. Babuška, J.M. Melenk, The partition of unity finite element method, Internat. J. Numer. Methods Engrg. 40 (1997) 727–758.
[6] I. Babuška, V. Nistor, N. Tarfulea, Generalized finite element method for second-order elliptic operators with Dirichlet boundary conditions, J. Comput. Appl. Math. 218 (2008) 175–183.
[7] R.E. Bank, L.R. Scott, On the conditioning of finite element equations with highly refined meshes, SIAM J. Numer. Anal. 26 (1989) 1383–1394.
[8] S.C. Brenner, L.R. Scott, The Mathematical Theory of Finite Element Methods, second edition, Texts in Applied Mathematics, vol. 15, Springer-Verlag, New York, 2002.
[9] P. Ciarlet, The Finite Element Method for Elliptic Problems, Studies in Mathematics and Its Applications, vol. 4, North-Holland, Amsterdam, 1978.
[10] W. Dahmen, S. Dekel, P. Petrushev, Multilevel preconditioning for partition of unity methods: some analytic concepts, Numer. Math. 107 (2007) 503–532.
[11] S. De, K. Bathe, The method of finite spheres with improved numerical integration, in: First European Conference on Computational Mechanics, Munich, 1999, Comput. Structures 79 (2001) 2183–2196.
[12] C.A. Duarte, J.T. Oden, An h–p adaptive method using clouds, Comput. Methods Appl. Mech. Engrg. 139 (1996) 237–262.
[13] C.A. Duarte, J.T. Oden, H–p clouds—an h–p meshless method, Numer. Methods Partial Differential Equations 12 (1996) 673–705.
[14] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
[15] M. Griebel, M.A. Schweitzer, A particle-partition of unity method. III. A multilevel solver, SIAM J. Sci. Comput. 24 (2002) 377–409 (electronic).
[16] M. Griebel, M.A. Schweitzer, A particle-partition of unity method. IV. Parallelization, in: Meshfree Methods for Partial Differential Equations, Bonn, 2001, in: Lect. Notes Comput. Sci. Eng., vol. 26, Springer, Berlin, 2003, pp. 161–192.
[17] M. Griebel, M.A. Schweitzer, A particle-partition of unity method. V. Boundary conditions, in: Geometric Analysis and Nonlinear Partial Differential Equations, Springer, Berlin, 2003, pp. 519–542.
[18] Y. Huang, J. Xu, A conforming finite element method for overlapping and nonmatching grids, Math. Comp. 72 (2003) 1057–1066 (electronic).
[19] S. Li, W.K. Liu, Meshfree Particle Methods, Springer-Verlag, Berlin, 2004.
[20] J.M. Melenk, On generalized finite element methods, PhD thesis, University of Maryland, College Park, MD, 1995.
[21] J.M. Melenk, I. Babuška, The partition of unity finite element method: basic theory and applications, Comput. Methods Appl. Mech. Engrg. 139 (1996) 289–314.
[22] J.T. Oden, C.A.M. Duarte, O.C. Zienkiewicz, A new cloud-based hp finite element method, Comput. Methods Appl. Mech. Engrg. 153 (1998) 117–126.
[23] J. Pereira, C. Duarte, Extraction of stress intensity factors from generalized finite element solutions, Eng. Anal. Bound. Elem. 29 (2005) 397–413.
[24] T. Strouboulis, I. Babuška, K. Copps, The design and analysis of the generalized finite element method, Comput. Methods Appl. Mech. Engrg. 181 (2000) 43–69.
[25] T. Strouboulis, K. Copps, I. Babuška, The generalized finite element method: an example of its implementation and illustration of its performance, Internat. J. Numer. Methods Engrg. 47 (2000) 1401–1417.
[26] T. Strouboulis, L. Zhang, I. Babuška, Assessment of the cost and accuracy of the generalized FEM, Internat. J. Numer. Methods Engrg. 69 (2007) 250–283.
[27] R. Tian, G. Yagawa, H. Terasaka, Linear dependence problems of partition of unity-based generalized FEMs, Comput. Methods Appl. Mech. Engrg. 195 (2006) 4768–4782.
[28] W. Wendland, G. Hsiao, Boundary Integral Equations, Springer-Verlag, New York, 2008.
[29] J. Xu, Multilevel Finite Element Theory, 2003.
[30] J. Xu, L.T. Zikatanov, On multigrid methods for generalized finite element methods, in: Meshfree Methods for Partial Differential Equations, Bonn, 2001, in: Lect. Notes Comput. Sci. Eng., vol. 26, Springer, Berlin, 2003, pp. 401–418.